LEARNING CREATIVELY WITH GIANT TRIANGLES

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In a mathematics methods course for pre-service teachers, carefully designed activities made deep and interconnected mathematics quickly available requiring no pre-requisite content knowledge. Using brightly colored 1-meter edge length equilateral triangles that can be quickly assembled and reassembled into a wide range of polyhedra by small groups of students, high levels of student engagement and collaboration were achieved. The van Hiele (1976) Model of Geometric Thought and NCTM (2000) process standards were explicitly referenced in the course.

INTRODUCTION

The challenge of developing deep content knowledge and pedagogical content knowledge (e.g., Shulman, 1986; Ball & Bass, 2000) for both in-service teachers and pre-service teacher candidates has been expressed for some time. We have achieved this, to new levels, with a unique inquiry experience to develop visualization and deep, analytical content knowledge as well as pedagogical knowledge within an urban pre-teacher mathematics methods course. The triangles that facilitated this are equilateral, 1-meter edge length, resilient, lightweight and brightly colored, and can be quickly assembled and reassembled into a wide range of polyhedra.

THEORETICAL FRAMEWORKS

Mathematics embodied in the manipulatives

The hidden mathematics framework suggested by Abramovich and Brouwer (2006) entails finding, creating and working with mathematical problems that connect across the mathematics curriculum. This helps prospective teachers make relevant connections between their undergraduate mathematics courses and the K-12 school curriculum. Their research entails integration of rigorous mathematics activity with technology-assisted learning in expert-novice, socially mediated classroom settings (Vygotsky, 1986). We claim that a similar learning environment can be fostered through carefully applied manipulatives such as the giant triangles. Furthermore, the triangles relate physically to learners through their kinesthetic character and aesthetic appeal. The giant triangles, as constructed, have powerful inbuilt mathematics that has to emerge when learners interact with them using very carefully designed activities. The learners' use of the materials changes as their knowledge develops observably shifting from concrete to abstract through interaction with the materials (M. L. Connell, personal communication, December 6, 2010).

van Hiele Model of Geometric Thought

The van Hiele Model of Geometric Thought (van Hiele, 1976), in differentiating increasingly complex levels of geometric thinking, is a useful framework to describe the evolving shape-building activities in this study. Additionally, its understanding was an explicit pedagogical goal for the pre-service teachers in the course. Reflection on the activities contributed to learners' pedagogical appreciation of the model.

Our activities are introduced at the Visual Level (What do you notice?) but quickly move learners to the Descriptive/Analytical Level (What properties can you specify? For example, how many vertices, edges and faces does this particular figure have?) and Relational Level (Which figure has this many faces? Or, How is this structure like that structure?), in which learners begin to abstract and generalize. No prior knowledge or experience is needed to engage in these activities, which reveal and develop deep mathematics without interference from learners' prior mis- or pre-conceptions. Learners collaborate and cooperate spontaneously in building their figures, facilitated by the size and construction of the triangles.

METHOD AND CONTEXT

University and course context

The university, one of the most ethnically diverse liberal arts institutions in the mid-south-western United States, is a federally designated Minority Serving Institution. It provides 4-year degree programs and has an open enrolment policy. A large percentage of its undergraduate students are the first college attendees in their families, and work full-time while attending college. The teacher certification/degree program requires students to take at least two mathematics content courses for teaching prior to their mathematics methods course, which is typically offered during their third year of study. Many students take these content courses at collaborating community colleges and then transfer to the university to complete their bachelors' degrees. The content courses tend to be factual and non-exploratory in structure.

The pre-service teacher methods courses are limited to no more than 30 participants and meet face-to-face for 2.5 hours per week over 14-15 weeks. While the curriculum for the methods courses spans all mathematics content strands, the geometry strand focuses on understanding the van Hiele model through interactive experience, integration of the National Council of Teachers of Mathematics (NCTM, 2000) process standards, and connections among the mathematics content strands.

Research method

The study is guided by the following research questions:
1) How do the triangle activities impact learners?
2) How do learners relate the triangle experiences to the van Hiele model?

Data consist of instructor field notes, students' online discussion comments, and photographs. A narrative approach is utilized to illuminate the findings.

LEARNING TRAJECTORY

We share a learning trajectory that was successfully enacted in ten mathematics methods classes for elementary and for middle-grades pre-service teachers in single 160-minute class sessions during two successive semesters.

Deepening the mathematics through pyramid construction

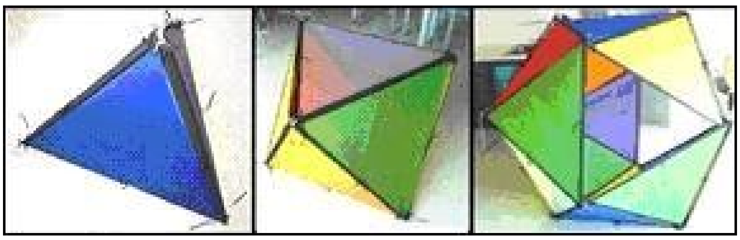
The trajectory began with a simple visual level activity: How many triangles can one fit around a given point laying flat on the floor? Learners predict and then build the figure. What happens if successive triangles are removed and the newly exposed sides are connected (see Figure 1)?

Figure 1. Triangles around a point and the three folded pyramids

This activity integrated a review of naming conventions for polygons and properties of the particular polygons and pyramids that emerged from this construction. Learners considered which pyramids might have the greatest and least volumes given that their base areas increased while their altitudes decreased with base side number (from 3 to 5). From a pedagogical perspective this problem was pointed out to be an alternate learning trajectory that could be enacted at this point in the lesson.

From pyramids to Platonic solids – Visual to descriptive level transition

Learners' next task was to attach additional triangles to the pyramids to form figures with the same number of regular triangular faces at each vertex. Figure 2a shows the three regular polyhedra that can be constructed using equilateral triangles. Initially, group members paid close attention to the number of faces at particular vertices but did not notice incongruence occurring at other vertices as they added triangles to the growing figure. Figure 2b shows some typical intermediate figures that learners constructed as they worked toward building the regular icosahedron. As the incongruent vertices were pointed out, learners began to attend more closely to properties of the figure, i.e., to the number of faces at every single vertex, rather than to its global appearance. In this way, the manipulative itself carried the mathematical development initially at the visual level toward the descriptive level.



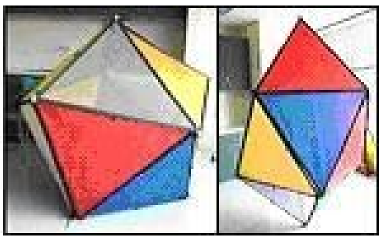


Figure 2a. Three regular polyhedra Figure 2b. Intermediate figures

Learners shared their insights facilitated by key questions dealing with comparing and contrasting the three regular polyhedra. These included visualizing the figures from different perspectives, such as when lying on a face versus standing upright on a vertex (Figure 3). The light weight of the manipulative allows one to lift such very large figures into the air so that one can actually stand inside them. A triangle was deliberately removed from the icosahedron (Figure 2a) so that each person could experience being inside it. This is a unique mathematical and aesthetic experience that can rarely be made available with other manipulatives.

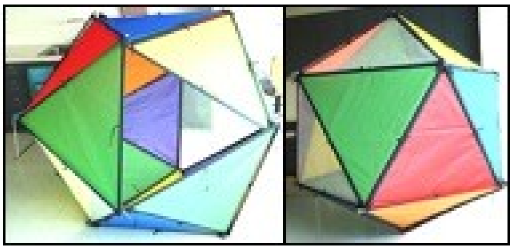
 

Figure 3. Lying on a face versus standing upright on a vertex

Symmetry and enumeration – Descriptive to relational level transition

Learners used the symmetries in the upright octahedron and icosahedron to justify their enumeration of faces. Enumeration of vertices and edges also evolved from symmetry considerations.

Generalization using other regular polygons – Relational level development

To further develop the entire visualization and enumeration process, the instructor asked the class to consider what would happen if only 2 triangles were connected. Then, what can be constructed using other regular polygons, such as squares, pentagons, hexagons, heptagons, etc. Only two new regular polyhedra, the cube and the dodecahedron, could be constructed.

Number patterns embodied in the Platonic solids

Using Polydrons™, individual learners constructed the cube and dodecahedron and then the whole class worked to complete the enumeration table as shown in Table 1. Symmetry again played an important role enumerating faces, vertices and edges for the new figures. Field notes from one session indicated that three particular patterns were discovered:

1) As listed in Table 1, as the number of faces increases, the number of vertices and edges also tend to increase;
2) the numbers are all even;
3) the numbers of vertices and faces switch for the cube and octahedron, and for the dodecahedron and icosahedron, while their edge numbers are the same; and,
4) the tetrahedron does not have a switch partner since it has the same number of faces as vertices.

These relationships were typical of those emerging across all classes. The concept of duality was revisited geometrically later in the lesson.

Name	Vertices	Edges	Faces
Tetrahedron	4	6	4
Hexahedron (Cube)	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Table 1. Enumeration of vertices, edges and faces

Doubling the tetrahedron – Descriptive and relational levels

The next challenge involved building a tetrahedron with doubled edge lengths. They noticed that a useful net for building a smaller, unit-sized tetrahedron was a larger, doubled edge length triangle consisting of 4 unit triangles. Therefore they would need four of these for the larger tetrahedron, 16 unit triangles in all. Scaling the lengths by a factor of 2 implies a scale factor of 2^2 for a "doubled" figure's surface area. For this figure, the net itself provided an opportunity to examine the sum-of-odd-numbers series shown in Figure 4.

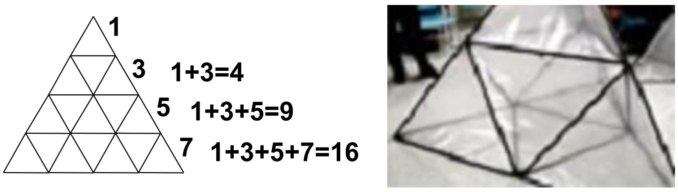


Figure 4. Doubling the edge lengths of the basic tetrahedron

Next, they were asked to predict how many unit-size tetrahedra would fill the larger tetrahedron. Prediction activities have been documented to enhance mental visualization (e.g., Battista, 1999). Learners predicted 4, 5, 6, or 7 unit-size tetrahedra would be needed to solve this problem, the majority choice being 5. The durability and construction of the triangles provides opportunities to examine these types of figure in novel ways, such as from the inside, that most other manipulatives cannot provide. Learners initially inserted four smaller tetrahedra into the spaces by the vertices of the larger figure. Then, they attempted to fill the center region with additional tetrahedra. Eventually someone realized that tetrahedra cannot tessellate this space. Some classes built the figure that fills the space by inserting loose triangles and connecting them together in place. Others recognized that the triangular base on the floor was rotated 180 degrees relative to the triangular base at the figure's midlevel, a property of the octahedron. The octahedral skeleton is visually evident in the photograph in Figure 4. At this point, the instructor referred back to the scale factor concept, that doubling the edge lengths should result in an 8-fold scaling of volume. Thus, the larger tetrahedron consisted of 4 smaller tetrahedra and 1 octahedron, which itself occupies a volume equivalent to 4 smaller tetrahedra. Referring back to the earlier problem of comparing pyramid volumes, the octahedron's volume is equivalent to the combined volumes of 2 square pyramids (see Figure 3). Therefore, the square pyramid's volume is equivalent to the combined volumes of 2 smaller tetrahedra. A different learning trajectory can be enacted to confirm this finding. Interconnections across the mathematics curriculum were constantly becoming apparent as this trajectory progressed.

Platonic solids duality exposed – Relational level

The final activity involved attaching small tetrahedra to the central triangles on the doubled-edge length tetrahedron's faces. When stood upright on one of its vertices this figure looked like two intersecting large tetrahedra, one pointing upward and one pointing downward. Learners used masking tape to connect the vertices of this stellated figure (see Figure 5). To their surprise, the tape formed the edges of a cube. Knowing that the interior of this figure was the octahedron (from the previous filling activity), the class was able to see that the centers of each face of the cube were the vertices of the interior octahedron. Referring back to Table 1, the switching of the vertices and faces of the cube-octahedron duals now made sense.

Figure 5. Stellated octahedron

The class then viewed a very short animated video clip on the stellated octahedron that exemplifies the duality properties of the cube and octahedron.

Learner reflections

A selection of unedited learner quotations, from online reflection discussions, shows a high level of learner engagement, collaboration, deeper thinking and understanding:

> I really enjoyed that the class worked together to figure the lesson out. It was not a teacher lecturing and the students un-engaged.
>
> It had the whole class involved. It seemed to get more input from individuals whereas if we were all sitting down just a few people would have responded.
>
> I got to walk inside the icosahedron! I think children and young adults would love this experience because it is basically playing while learning.
>
> The value of this activity is not only that it is hands-on but that it also reaches beyond the surface of just looking and playing, the class could explore in depth.
>
> I actually got the concept we were learning from the big triangles. I'm not sure if it is because it was more hands on or because I was working with my peers, and they helped me understand.
>
> On the day this lesson took place I had no idea how many aspects of education it was going to cover. I was also very surprised on how large this project was, this was not a bad thing it played an important role in what was being taught and how. Not only was this project informative on how to teach our future students but it was full of application of how to teach in large groups. So looking back and taking in what this lesson was about it was about math, but it was about so much more.
>
> … it shows how if you have a flat shape and you take away a triangle it can turn it in to a

> polyhedron. It was amazing just by manipulating a few triangles could completely change the properties of a shape.
>
> This project also can help create a learning community because this cannot be done without team work. All of the students have to pitch in or else the lesson will not reach its full potential.

Many students mentioned that the lesson was fun. This suggests that the hands on play aspect of the giant triangles and collaboration they engender have an affective advantage associated with successful learning in carefully planned activities.

CONCLUSIONS

Developing deep content knowledge and pedagogical content knowledge for pre-service teacher preparation can be challenging. Several important points about how we achieved this development to new levels with this trajectory include:

1) Substantial, deep, and interconnected mathematics, as described by Abramovich and Brouwer (2006) is made available quickly and effectively using the triangles. These activities reinforce the NCTM (2000) process standards of communication, problem solving, connections, reasoning and justification, and representation while interconnecting with other content strands (measurement and algebraic thinking).

2) No entry-level content knowledge is required and transfer from prior content courses has generally not been observed. In attempting to bring highly interactive and interconnected mathematical experiences to our methods classes, a big challenge is to overcome learners' attitudes about this intense form of teaching since most learned to "do math" in very traditional "copy the model and practice" ways. Thus, weakly conceptualized mathematical knowledge may intrude on the learner's openness toward deeper mathematical understanding and pedagogy. However, the triangles immediately engage learners, who remain open-minded to the mathematics and to the methods as new activities are introduced.

3) High levels of student engagement and collaboration are achieved associated with hands on play and figuring out activities, in a positive affective social context. For example learners did not give up or express frustration or discouragement from making 'intermediate figures' that needed correcting as in Figure 2b. Rather, they enjoyed the teamwork required to complete the activity. This shows how the use of the triangles facilitates social mediation of mathematical thinking by requiring negotiation of co-operative actions through visualization. Such negotiation promotes communication at deeper levels of geometric thought, even though the learning objectives were relatively simple (e.g., Add more triangles to the figure so that each vertex contains the same number of triangular faces.)

4) Use of these manipulatives may avoid some of the affective pitfalls that occur when introducing challenging mathematical problems. Researchers have noted the importance of 'the struggle' and often-associated 'perplexity' when extending one's mathematical knowledge through difficult problems (Hiebert & Grouws 2007). With

this trajectory, we believe that the aesthetic and size appeal of the triangles, and the personal commitment to the collaborative constructions, enabled learners to persist, working in groups, until achieving success. At no point did they give up. Even during the enumeration activity, relying on mental imagery, all learners participated with enthusiasm transferring the knowledge just gained though the hands-on experience.

Future work

These activities will be placed earlier in the course to help open learners' minds to deep mathematics and conceptual learning methods. Future learners will write additional reflections on how the different van Hiele levels were addressed across all course-based geometry activities. Also, work has begun with a relatively new teacher in a struggling urban middle school. Her 6th graders have mainly had skill-driven mathematics experiences aimed to raise test scores. The triangles have brought them enthusiasm through an initial activity, to 'build something interesting and beautiful'. We will continue to develop their mathematical knowledge and dispositions toward doing mathematics, while co-teaching with their teacher to develop her pedagogy.

Acknowledgement

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