11 Faces and Edges

Themes	Number patterns, graphing, linear equations, definition and reasoning.
Vocabulary	Face, Edge, Vertex, Proportional, Ratio, Slope, Plane, Coplanar, Polygon, Polyhedron, Tetrahedron, Restricted Domain.
Synopsis	Count the numbers of faces and edges on a range of polyhedra. Tabulate and graph results, look for patterns in the table and graph and express as an equation, determining the ratio of faces to edges, and reasons for the ratio. Examine how varying definitions of face can give varying results. For advanced students, discuss different types of proof.

Overall structure	**Previous**	**Extension**
1 Use, Safety and the Rhombus (required)	**X**	
2 Strips and Tunnels extends basic building and shape		
3 Pyramids (if not already done)		**X**
4 Regular Polyhedra (if not already done introduces the Euler formula which relates numbers of faces, edges and vertices)		**X**
5 Symmetry		
6 Colour Patterns		
7 Space Fillers		
8 Double edge length tetrahedron		
9 Stella Octangula		
10 Stellated Polyhedra and Duality		
11 Faces and Edges		
12 Angle Deficit (also looks at patterns in numbers associated with polyhedra)		**X**
13 Torus		

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 Introduction and prediction

Hold up a triangle and ask:

How many triangles do you think or guess are the least number needed to make a closed up 3 dimensional shape?

4, '5'

Break the class into groups of four or five students for each group to make a closed up shape with the minimum number of triangles. Give the instruction:

Each group make a closed up shape with as few triangles as you can.

After they have had time to build tetrahedra, ask:

How many triangles did you need?

4

2 Terminology

Discuss the terms 'side', 'edge', 'vertex', 'face', 'polygon' and 'polyhedron' according to definitions in your curriculum, for example:

A polygon is a two-dimensional shape with straight sides.

What is a 'polyhedron'?

A 3-D shape with no curves

A polygon has sides and vertices. What does a polyhedron have?

Faces, edges, vertices, 'sides'

We only use the term sides for polygons, we use edges and faces for polyhedra.

This can lead into a discussion of the ambiguity of the term 'side' for 3 dimensional objects. The 'side of a house' could be a wall, whereas the 'side of a tetrahedron' could be an edge.

Hold up one of the tetrahedra and ask:

What kind of faces does this polyhedron have?

Triangles

If you have a cube hold it up and ask:

Is this a polyhedron?

Yes

What kind?

Cube

Hold up a box and ask:

Is this box a polyhedron?

Yes

What kinds of faces does it have?

Rectangles

Define 'polyhedron' according to your curriculum, for example:

So a polyhedron is a 3 D shape whose faces are all polygons. These polygons meet along their sides without gaps or overlaps, forming edges of the polyhedron.

3) Counting faces and edges

Hold up a tetrahedron and ask:

How many faces and edges does this shape have?

4 faces, 6 edges

Show us

One here, two more here, and here … etc.

It may help to lay the shape on the ground if the students' explanations are not clear, and go over the counting. Touch the faces and edges as you go.

We can see 3 faces sloping up which I can touch, here, here and here, and one underneath. So that is 4 faces. There are 3 edges on the ground which I can touch here, here and here, and 3 going up to the apex, here, here and here. That is 6 edges.

4 Naming the tetrahedron

Ask:

What do we call this shape?

Tetrahedron, pyramid

Why do you say tetrahedron?

It has four sides

What is the term we use here with polyhedra instead of 'side'?

Faces

We can say it is a tetrahedron because it has four faces, ('tetra-' means 4 and '-hedron' means face or head)

Why do you say pyramid?

It comes up to a point

Both names are correct, a tetrahedron because it has four faces, and it is also a triangular based pyramid.

5 Table and graph

Draw on the board, or chart paper, a table for the class with column headings, 'Faces' and 'Edges', see figure 1. Enter in the first column the numbers of faces from 4 to 16. Enter the results from each group for the number of edges (6) on a polyhedron with four faces in the first row of the second column.

Faces	Edges
4	6,6,6
5	
6	9,9,12*
7	11*
8	12,12
9	
10	15
11	
12	18,18
13	
14	21
15	

Figure 1 Completed table with all entries (*shows anomalies)

Tell the students:

> You are going to build different shapes and count the number of faces and edges.

For basic level students add the specific instruction:

> To count the number of faces, count the number of triangles.

Then for all students say:

> Each group make a copy of the table on paper. First enter your results as you get them there. Then as you get each result send someone up to fill it in on the board.

Also now, or after all results are in, make a corresponding graph for the class. You may choose to use software, some chart paper or a board. Allow Faces (F) to range from 0 to 16, and Edges (E) from 0 to 25, and explain that this allows for more edges than faces.

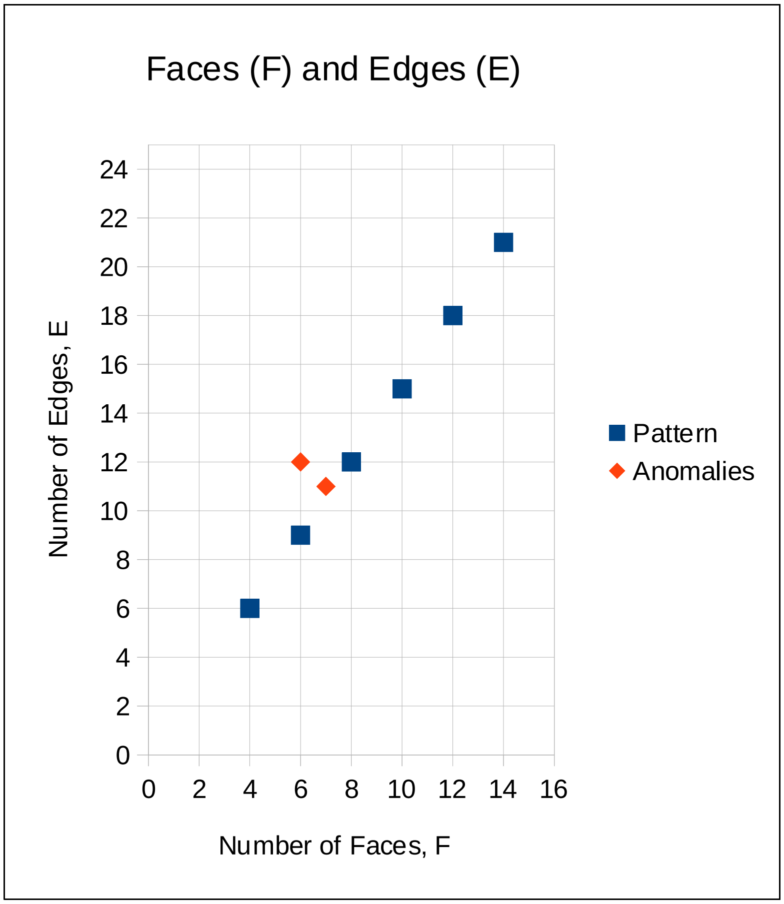

Figure 2 Graph showing pattern and anomalies

If you do this before the building work, add the instruction:

> Also make a copy of the graph and plot your values on the graph as you get results, and also add the point you plot to the graph on the board.

This may show mistakes plotting but also anomalous results that can lead to a discussion of the definition of a face. See section 7 below.

6 Building, recording and looking for patterns

Ask groups to:

> Continue building shapes for different numbers of faces, without untying polyhedra made, so they are saved for class discussion.

They may not be able to build shapes for all the rows of the table. Keep asking the individual groups

Have you been able to make polyhedra for all numbers of faces?

Can you see any patterns in the numbers or in the graph?

Asking groups individually will mean more thinking and discussion may take place in groups. This may enable a richer class discussion. Many groups will say:

There has to be an even number of triangles

Once sufficient results are gathered, and **before** groups untie shapes, have a class discussion on patterns and findings. Write down suggestions for results, such as:

'E goes up by 3 and F goes up by 2'

'Add half the number of faces to the original number of faces and that will be the number of edges'

This disucssion is developed further in section 8.

7 Finding and dealing with anomalies: Discussion of the definition of a face

Points that do not fit the line on the graph present an opportunity for discussion. At the end of this section we show how a polyhedron can be assembled that can illustrate different ways of counting faces and edges. These different ways give different results, and thus enable the following type to discussion to take place.

Ask if there are any anomalies in the results:

Are there any results that do not seem to fit the rest?

They may come up with points (shown with * in the table in figure 1 and as series 2 on the graph in figure 2) that are not inline with other points. This may be because they are counting coplanar triangles, such as two forming a rhombus, as one face. Have the students see for themselves that some points on the graph or entries in the table do not follow the pattern but others do.

Ask all the groups to show the class how they counted faces and edges for the different shapes in question or that give different answers.

For the data filled in above this will be:

We got different answers for 6 faces. Who has a shape with six faces and show the class how you counted the edges?

Can anyone see the difference in ways of counting that cause the different answers?

Yes it depends on if a face can only be one triangle

Which other points do not seem to fit?

7 faces

Who has a shape with 7 faces? Show us how you counted the faces and edges.

This is one face with these two triangles because they are flat

If you counted each face as being a triangle, how many faces do you get?

8 faces

We will mark separately these shape's numbers and points on the graph which were counted with more than one triangle making a face.

You can circle the values in the table and graph that have been identified as anomalous. If instead you used software, you can form separate plots as in figure 2.

What can you say about the graphs and table now?

'All the remaining points lie on the line'

'All the remaining numbers in the table fit the pattern'

Summarize the effect of choice of definition:

If we define a face as a triangle then all the data follows the pattern and fits on the line, otherwise it does not.

This gives an opportunity to introduce vocabulary:

We use the word 'collinear' to mean lying on the same line, so all the points on the graph that lie on the line are collinear.

We use a similar word 'coplanar' to mean lying in the same plane, so two triangles that make up a flat rhombus are coplanar.

The above discussion needs to take place before the shapes are untied.

If no anomalous results occur, or no coplanar faces emerge from the activity, a shape can be built as follows with coplanar faces.

Figure 3 Making a polyhedron with coplanar triangles

Figure 3 shows a polyhedron made by placing an octahedron and tetrahedron together, face to face. It has three pairs of coplanar triangles. Each pair has one red and one clear triangle which form a rhombus. If each rhombus is counted as a face, then we can say the polyhedron has 7 faces and 12 edges. Alternatively if only triangles are considered to be faces then we have 10 faces and 15 edges.

8 Number patterns, equations and slope

Cover as much of these discussion points as is appropriate for your class.

- Try to have students express any number patterns they found as algebraic relations
- Obtain the equation $E=3F/2$

- Connect it to the slope 3/2 of the graph
- Comment on how the line through points goes through the origin and discuss proportionality

9 Asking if it is always true?

Many students will probably already have claimed that you can't make a closed up shape with an odd number of triangles. This gives an opportunity for a discussion of justification or even proof.

We found we can't make a closed up shape with an odd number of triangles. Is this always true, even if we try with many more triangles, like 99 or 9999? How do we know if we have not tried?

To add more triangles you open up along two sides and add two more triangles, so you are always adding two more faces. So it has to be even

Can you show a polyhedron where you did this, or show us what you mean by doing it to one of your shapes?

Like this

So you are stellating one face which you can also think of as adding a pyramid to the face, or replacing one face with a pyramid to make the new polyhedron.

You showed that when you start with a tetrahedron you can keep adding 2 more triangles to make another closed up shape. So you can have any even number bigger than 4.

We see that E =3F/2. Why is it 3F/2, can you explain that? Is there a reason for it?

When you add two more triangles you add three more edges

That shows where the 'multiply by 3' and 'divide by 2' come from, for every 2 triangles you add, you have to add 3 edges. That works because you start with the tetrahedron which has 4 faces and 6 edges.

The above replies are real student replies. See the next section for a simple explanation.

10 An explanation of why it is always true

Here is an explanation you can give students why an even number of triangles is needed.

Hold up a triangle and say:

Inside each edge here is a rigid rod.

How many rods in the triangle?

3

Pick up a polyhedron with for example 10 faces

How many faces does this polyhedron have?

10

If there are 10 triangles in this polyhedron, how many rods will there be in total?

30

Now hold one edge of the shape in your hand clutching both sides of the edge in one hand and say

How many edges am I holding in this hand?

1

And how many rods am I holding in this hand?

2

Each edge has two rods, and there are 30 rods, so how many edges?

15

Why?

'have to divide by two because it is two rods for each edge'
'There is only half the number of rods as edges'

Yes there are twice as many rods as edges, because there are two rods in each edge, so that means there are half as many edges as there are rods.

So why is E=3F/2?

Because you have to multiply by 3 and then halve it

Yes, you have to multiply by 3 to get the number of rods and divide by 2 to halve it for the number of edges.

Also we learned that we can't have an odd number of faces because 3/2 times an odd number is not a whole number. So if we had a shape with 5 faces, how many edges would that be?

7 and a half

Yes 3 times 5 is 15, then divide by 2 gives 7 and a half. Is that possible, to have half and edge?

No

11 Footnote on proofs for activity with advanced students

The suggestion by students in section 9 that you can add two triangles at a time (i.e. stellating a triangular face) is an attempt to prove by induction that you can always make a shape with any even number of triangles greater than or equal to 4. It does however fail to prove the converse that it is impossible to make a closed up shape with an odd number of triangles. The explanation in section 10, however, is a proof. It uses a counting argument rather than an induction argument.