12 Angle Deficit

Themes	Angle measure, polyhedra, number patterns, Euler Formula.
Vocabulary	Vertex, angle, pentagon, pentagonal, hexagon, pyramid, dipyramid, base, degrees, radians.
Synopsis	Define and record angle deficit at a vertex. Investigate the sum of angle deficits over all vertices in a polyhedron. Discover its constant value for a range of polyhedra.

Overall structure	**Previous**	**Extension**
1 Use, Safety and the Rhombus		
2 Strips and Tunnels		
3 Pyramids (gives an introductory experience of removing angle at a vertex to make a three dimensional object)	X	
4 Regular Polyhedra (extend activity 3)	X	
5 Symmetry		
6 Colour Patterns		
7 Space Fillers		
8 Double edge length tetrahedron		
9 Stella Octangula		
10 Stellated Polyhedra and Duality		
11 Faces and Edges		
12 Angle Deficit		
13 Torus (extends both angle deficit and the Euler formula. For advanced level discussion see the separate document: "Proof of the Angle Deficit Formula")		X

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 Quantifying angle deficit

Ask the students:

How many of these equilateral triangles can fit around a point flat on the floor, and what shape will they make?

6, hexagon

Get into groups of 4 or 5 students and take enough triangles to lay them out around a point flat on the floor and tie them.

How many did it take?

6

What shape is it?

Hexagon

Why is it called a hexagon?

It has 6 sides

Take two hexagons and put them near each other but not touching, and untie one triangle from one of the hexagons as in figure 1. Ask:

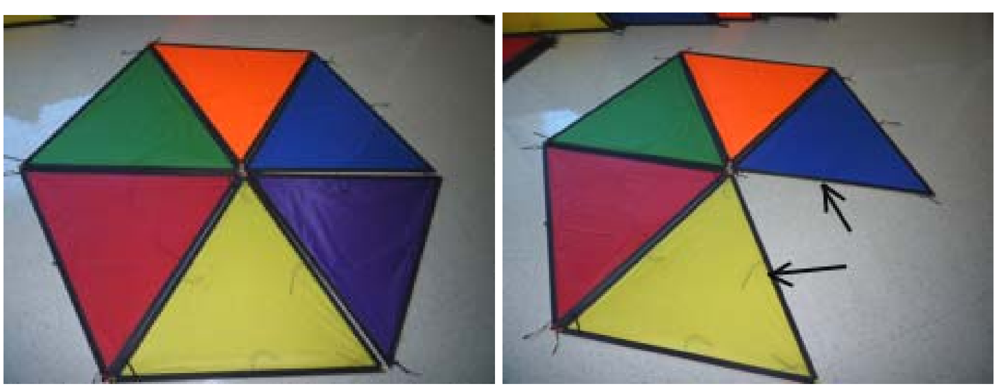

Figure 1 Six and five triangles at a point

How many triangles are left?

5

How much did I take away?

One triangle

What fraction of all 6 is that?

1/6

Talk in terms of degrees and/or Radians depending on the class.

How many degrees {or Radians}?

60

{π/3}

How many degrees {or Radians} in a whole turn all the way round a circle?

360

{2π}

What fraction of 360 degrees {or 2π Radians} is 60 degrees/radians?

1/6

Now we will enter these numbers on a table with columns for degrees {and radians}

Start filling out a version the following table depending on which columns you wish to use.

No of Triangles at a point	No of triangles removed	Fraction of triangles or whole turn removed	Angle removed in degrees	Angle removed in Radians
6				
5	1	1/6	60	π/3

Now repeat the above questions after removing two and three triangles from two more hexagons respectively, and laying them all out as shown in figure 2. Complete the table below, with the columns you wish to use.

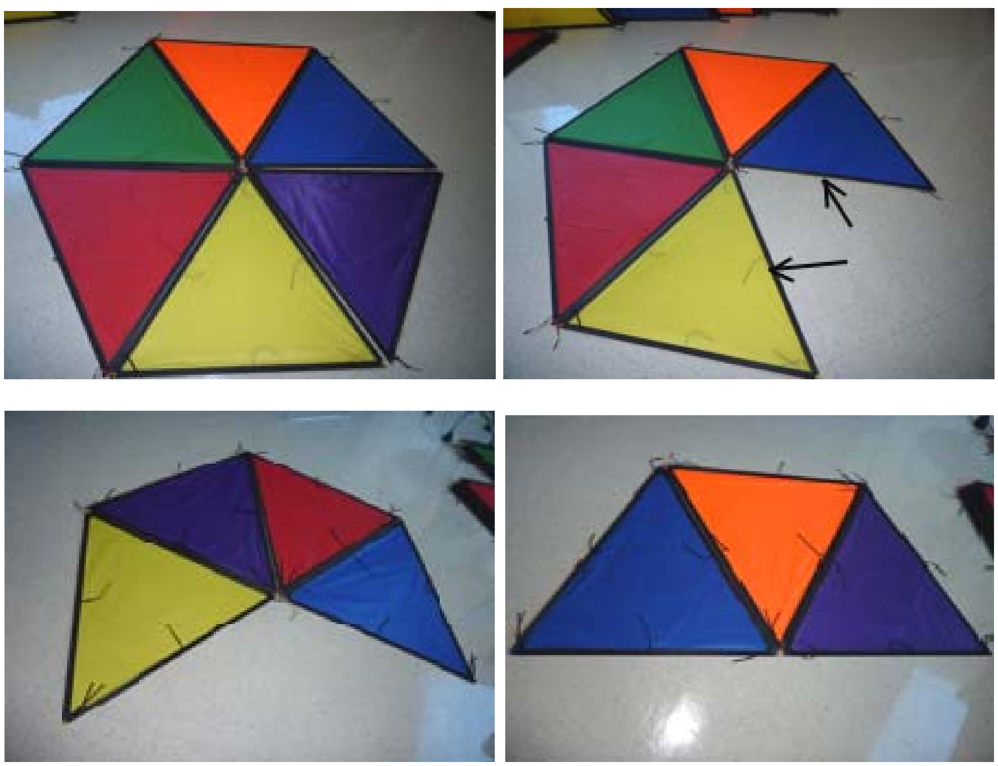

Figure 2 The removed angles

No of Triangles at a point	No of triangles removed	Fraction of triangles or whole turn removed	Angle removed in degrees	Angle removed in Radians
6	0	0	0	0
5	1	1/6	60	π/3
4	2	1/3	120	2π/3
3	3	1/2	180	π

2 Construct and qualitatively compare pyramids

Now have the students close up the gaps created by the removed triangles and tie the edges. See the edges marked by arrows in figures 1 and 2. This will give the pyramids as shown in figure 3

Figure 3 The vertices closed up

Line the pyramids in order of number of triangles removed to help show the trends. Ask:

What effect does removing more or less triangles have on the pyramid formed?

Steeper sides

Does removing more make the sides steeper or less steep?

More steep

What else does removing more triangles do?

Makes a higher pyramid

Anything else?

Make the top more pointed

Write up the results up at the front in a two column table such as:

Less Triangles/Angle Removed	More Triangles/Angle Removed
Flatter Bigger base area More faces More base sides	Higher apex Steeper sides Sharper point Less faces Less base sides

3 Setting up data collection

Explain:

> We are going to investigate the total amount of angle removed from all the vertices in a polyhedron.
>
> We are going to use a table for each shape to do it systematically, but we need to start with an interesting shape to do this.
>
> Make another pentagonal pyramid, and tie the two pentagonal based pyramids together base to base.

Once students have built this shape, a pentagonal dipyramid, continue with the explanation:

> Now we will work out how to fill in the table row by row

You will need to decide how you are going to sum up the angle deficit by choosing one or more of the tables below to work with, depending on which units to count in with your students:

- Number of triangles removed (from 6 triangles) at vertices
- Fraction of a whole turn removed (from 1 whole turn)
- Angle in degrees removed (from 360 degrees)
- Angle in radians removed (from 2π Radians)

The table or tables you choose can be printed and handed out. One copy of a table is needed for one polyhedron.

Measuring in number of triangles removed:

Triangles at vertex	5	4	3	Total
Number of vertices				
X Number of triangles removed at one vertex	1	2	3	-
= Number of triangles removed over set of vertices				

Measuring in fraction of whole turn removed:

Triangles at vertex	5	4	3	Total
Number of vertices				
X Fraction of whole turn removed at one vertex	1/6	1/3	1/2	-
= Fraction of whole turn removed over set of vertices				

Measuring in degrees removed

Triangles at vertex	5	4	3	Total
Number of vertices				
X Degrees removed at one vertex	60	120	180	-
= Degrees removed over set of vertices				

Measuring in radians removed

Triangles at vertex	5	4	3	Total
Number of vertices				
X Radians removed at one vertex	π/3	π/2	π	-
= Radians removed over set of vertices				

We now illustrate filling in a table using the unit 'fractions of a whole turn'.

Ask:

What shall we call our polyhedron?

'Flying saucer'
'Spinning top'
'A pentagonal diamond'
'A double pyramid'

It is called a 'pentagonal dipyramid' which means means two pentagonal based pyramids connected together

Hold the dipyramid with a 5-triangle vertex on the floor. Ask:

What kinds of vertices are there and how many triangles does each kind of vertex have?

Top and bottom and
Round the middle.

How many triangles on the top vertex?

5

How many on the bottom?

5

How many at each middle vertex?

4

How many vertices are there round the middle?

5

Now return to the table and fill in:

There were 2 vertices, the top and bottom with 5 triangles, so we put 2 here, for number of vertices with 5 triangles.

How many with 4 triangles at a vertex?

5

So we put 5 here.

Any vertices with 3 triangles at a vertex?

No

So what should I write here?

0

How many vertices in total?

7

The table will now look like this:

Triangles at vertex	5	4	3	Total
Number of vertices	2	5	0	7
X Fraction of whole turn removed at one vertex	1/6	1/3	1/2	-
= Fraction of whole turn removed over set of vertices				

Now we move to the bottom row.

So 2 vertices each have 1/6 of a whole turn missing, what does that add up to?

1/3

Yes, 2 times 1/6 is 1/3. We write in 1/3.

Now 5 vertices each have 1/3 of a whole turn removed, how much is that all together?

5/3

Yes, 5 times 1/3 is 5/3, so we write in 5/3.

What goes in the bottom of the next column?

0 times 1/2 is 0

Yes so we write in zero.

Now what is the final total?

6/3

What is another way to say that?

2

Yes 3 thirds is one, 6 thirds is 2, so we write 6/3 = 2 here.

Triangles at vertex	5	4	3	Total
Number of vertices	2	5	0	7
X Fraction of whole turn removed at one vertex	1/6	1/3	1/2	-
= Fraction of whole turn removed over set of vertices	1/3	5/3	0	6/3=2

Now build other polyhedra and fill in a table for each one and see what results you get.

Students may build elaborate polyhedra if they take lots of triangles, in which case they may ask:

What happens if there are six triangles at a vertex?

Can six triangles meeting at a vertex lay flat on the floor?

Yes

That counts as no deficit because it is completely flat.

If they ask:

What happens when there is more than 6 triangles?

Explain as follows:

If you have 7 triangles then you have an excess of 1 triangle. So we will call that a deficit of -1 triangle.

If you have 8 triangles then you have an excess of 2 triangles. So we will call that a deficit of -2 triangles.

A positive excess is a negative deficit.

4 Results

Students will nearly always find the sum of angle deficits is constant. If they do not then either they have made a mistake, or built a polyhedron with a hole through the middle, like a torus, or the polyhedron is not closed up. In each of the units, the constant values are a deficit of:

- 12 triangles
- 2 Whole turns
- 720 degrees
- 4π Radians

You all found the total angle deficit is constant. This constant stays the same for all convex polyhedra or even polyhedra that do not have holes going through them the way a doughnut has.

If a student asks if this 2 whole turns has anything to do with F-E+V, other than coincidence, it does, as explained in the separate document Proof of the Angle Deficit Theorem.

5 Prediction the number of vertices for regular solids.

Ask the question:

A mystery shape has 3 equilateral triangles at every vertex. The total angle deficit is the same as you found. What is the angle deficit at every vertex?

3 triangles

'½ a whole turn'

'180 degrees'

'π Radians'

So how many vertices in total?

4

So what is the method you used?

Divide the total angle deficit by the deficit at each vertex.

Yes, when all the vertices are the same we can write:

Write on the board:

Total number of vertices
=
(Total angle deficit) ÷ (Angle deficit at one vertex)

and that works because

Total angle deficit = (Total number of vertices) x (Angle deficit at one vertex)

This can be repeated for all the regular polyhedra, although the cube and dodecahedron cannot be counted in units of triangles missing at a vertex. Beyond that for advanced students, this method can also be applied the Archimedean solids which have fixed combinations of regular polyhedra at each vertex. For example two squares and two equilateral triangles at each vertex have an angle deficit of 60 degrees, and so there will be 12 vertices. This calculation applies both to a cube-octahedron and to a regular hexagonal prism.

6 Extensions

The paragraph above shows how advanced students could study the Archimedean solids.

As an extension you can ask:

What happens if you try to make seven or eight triangles come together at a vertex flat on the floor?

It won't go

Take triangles and try it

It bumps up.

It ruffles up

So extra triangles make it bump up or ruffle up, but where does it bump up or ruffle up?

on the outside,

and less than six triangles make it bump up where?

on the inside.

And six triangles?

Don't bump up

Six triangles lie flat.

Then you can go into how to count angle deficit.

If you have 7 triangles then you have an excess of 1 triangle. So we will call that a deficit of -1 triangle.

If you have 8 triangles then you have an excess of 2 triangles. So we will call that a deficit of -2 triangles.

A positive excess is a negative deficit.

Now try to make a polyhedron with 7 or 8 triangles at a vertex, and see if this still gives the same constant angle deficit.

The activity of building a torus (**12 The Torus**) yields a different angle deficit total from the one found for simple polyhedra (2 whole turns, 720 degrees, etc.) and a value for F-E+V other than 2. This is because the torus has a hole going through it, and the Euler formula no longer applies.

In fact it is true for any polyhedron that F-E+V = total angle deficit over the polyhedron measured in whole turns. See separate document ‘Proof the angle deficit theorem’. Also the further reading contained within the document relates angle deficit to Gaussian curvature.

