13 Torus

Themes	Nets, geometric properties and features.
Vocabulary	Torus, trapezoid, convex, concave, annulus, genus.
Synopsis	Build and explore a torus, looking at symmetry, the inside and outside, and going round the inside tunnel. Compute total angle deficit and F-E+V, and find they are zero for a torus. Identify symmetries of the constructed torus.

Overall structure	**Previous**	**Basic Extension**	**Advanced Extension**
1 Use, Safety and the Rhombus			
2 Strips and Tunnels			
3 Pyramids			
4 Regular Polyhedra (introduces the Euler Formula which is extended here)	**X**		
5 Symmetry (for intermediate level students OR elaborates on finding symmetries)	**X**		**X**
6 Colour Patterns			
7 Space Fillers			
8 Double edge length tetrahedron			
9 Stella Octangula			
10 Stellated Polyhedra and Duality			
11 Faces and Edges			
12 Angle Deficit (is extended here)	**X**		
13 Torus			

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 Assemble and place the net parts

Leading the students, assemble and tie the two parts of the net of the torus, as shown in figure 1. This needs a space that is large enough such as a lawn or gym. Note that it is important that the outside triangles of one part should be one colour, and the other part another colour. This way students can see at a glance which triangles are which during assembly.

Figure 1 Net of upper and lower parts of torus

Now slide one net over the other so they are centred, but at a different angle, where on is rotated 30 degrees, relative to the other. See figure 2. This will create a hexagonal hole in the centre of the assembly. Once you see the hexagon you have the parts well aligned.

Figure 2 upper net placed over lower net for tying (Note central hexagonal hole)

2 Joining the two parts of the net

Figure 3 The outermost single triangles of the upper net.

We are ready to start joining the two parts. First identify the three outermost single triangles on the upper part of the net. In figure 3 these are marked white discs on the far left, bottom right and right of top centre. The three form the apexes of the only edge length 4 triangle in the upper part of the net.

Figure 4 interleave a blue between two coplanar oranges to make a trapezoid/trapezium

Figures 4 and 5 show how to connect the two free edges of the outermost triangle of the upper net to triangles of the lower net. Note trapezoids are formed in this process.

Figure 5 Reverse side reveals a clear trapezoid.

Figure 6 shows this repeated with the other outermost triangles of the top part.

Figure 6 Complete the other two trapezoid tied structures

3 Completing the outer wall

Now continue constructing the outer wall of the torus. The zig-zag dogtooth pattern of one colour coming up from the bottom and another colour down from the top will continue all the way round the outer wall as shown in 7.

Figure 7 Continue tying around the outer ring of orange and blue zig zag pattern

4 Filling in the inner hole walls

All that remains is to seal off the hole in the middle. Again use a one colour up one down zig-zag pattern of 6 triangles as shown in figure 8.

Figure 8 The completed torus

5 Vocabulary and looking at the outside and inside

Once the torus is complete ask:

What observations can you make about this shape?

'Lots of flat faces', 'coplanar faces', 'Hole in the middle', 'concave'

These are all valid observations. We can add go through useful vocabulary here.

There are coplanar triangular faces. Who can run their hands over a set of coplanar triangular faces?

Yes, on top

Are there any more sets of coplanar triangular faces?

Yes the sides and underneath?

Notice that these questions we specify triangular faces.

What do we call the shape made by the coplanar triangles on top?

A ring

Some sketches on the board may help with the explanation of these definitions:

We call it an 'annulus'. An annulus can be a disc with a disc cut out of the middle, as long as it is completely cut out of the inside, and the cut out does not touch the edge. An annulus can be a triangle with a hole in the middle. It can a polygon with a polygonal hole inside, not touching the edge. Any two dimensional flat shape with a hole on the inside not touching the edge.

What objects do you know that are an annulus?

Washer

We also have a name for three dimensional shapes that have a hole or tunnel going through the middle. We call them a 'torus'. This shape we built is a torus and what objects do you know that is a torus?

'Doughnut', 'mug'.

For advanced students you can include:

Now there is also a name for the number of holes in a three dimensional shape. It is called the 'genus'. So this is a genus 1 torus, because it has one hole.

What objects can you think of that have genus, and what is the genus of the objects?

Roll of tape has genus 1.

Pair of scissors has genus 2

Figure 9 and 10 show how the torus can be seen. First through gaps and then standing on an edge.

Figure 9 looking inside through the gaps

Figure 10 The torus on its side

Figure 11 The torus inside and out

Finally allow students to go round the inside tunnel by going through the door and round the tunnel as in figure 11.

6 Euler's formula?

If students have already done Euler's formula in a previous activity then. Ask them

> Count faces, edges and vertices of all the triangles of the torus and compute F-E+V. What do you get?
>
> *We get zero.*

The counting is not trivial and they may need to go inside to check and put stickers in to be sure. This will take some time. However zero is the right answer because the torus (which has genus 1) has the hole going through it, and so Euler's formula where F-E+V=2 does not apply.

7 Angle deficit

If students have already done the angle deficit activity ask them

> Compute the angle deficit for all the vertices, and find the total
>
> *What happens when there are more than 6 triangles?*
>
> If you have 7 triangles then you have an excess of 1 triangle. So we will call that a deficit of -1 triangle.
>
> If you have 8 triangles then you have an excess of 2 triangles. So we will call that a deficit of -2 triangles.

A positive excess is a negative deficit.

So using that way of counting deficit, what is the total going to be?

The total comes out at zero.

What about faces minus edges plus vertices?

That comes out at zero too!

Yes both come out at zero because of the hole in the torus.

8 Symmetry

With the torus flat on the ground, ask

What are the symmetries of our torus?

Rotation 1/3
Reflection with a twist

What is the axis of the 1/3 turn rotation?

Vertical

What is the reflection with a twist? Show us.

Horizontal reflection, then twist it round to line up

So that is a combination of reflection and rotation.

Can anyone see any more?

Reflection in a vertical plane through the centers of the edges.

Show us one

How many vertical planes are there?

Three

Show us them

Can anyone see any more?

No

Figure 12 Holding a mid point

Now have one student hold the midpoint of the edge shown in figure 12 marked with a white dot. This is where two trapezoids share an edge.

Place other students around the torus but back at a distance, each one lined up with an edge between two trapezoids. Tell them.

Some of you are looking at a specific edge on the wall where two trapezoids meet and make a ridge. One of you is holding one of those edges in the middle.

Don't move from where you are standing and remember where the edge you are looking at is and which way it points. Up to the left or up to the right.

The one holding the edge do not let go as the remaining students not watching a specific edge lift and twist the torus round to flip it over upside down, then put it down where it is now, without flipping it back again.

Once this is done ask:

The person holding did you let go?

No

Are you in the same place you started?

Yes

Everyone, is your edge in the same place as it was and going up left or up right the same way it was?

Yes

What happened to the colours on the wall

They switched.

What transformation did we do?

Rotation

What kind?

½ turn about a horizontal axis.

9) Further extensions

The results change again if extra holes are added to increase the genus. This can be explored using wooden blocks with square faces. The results will be that if we measure angle deficit in numbers of whole turns,

Deficit in whole turns = F – E + V = 2 - 2(Number of holes)
= 2 - 2(genus) = The Euler Characteristic

Note the proof that the angle deficit is related to F – E + V is given in a separate document. Also, rather than using the term Euler's formula which only applies for genus 0, we now use the term The Euler Characteristic which is true in general.

