4 Regular Polyhedra

Themes	Pyramids, regular polyhedra, proof by demonstration, Euler's formula.
Vocabulary	Polyhedron, polyhedra, tetrahedron, octahedron, cube, icosahedron, dodecahedron, faces, sides, edges, vertices.
Synopsis	Extend pyramids with regular bases to form the regular Platonic polyhedra with triangular faces. Then find the other regular polyhedra and demonstrate there are only 5. Count faces, edges and vertices and discover the Euler formula. Note: Nets for polyhedra are intentionally **NOT** used in this activity.

Overall structure	**Previous**	**Extension**
1 Use, Safety and the Rhombus	**X**	
2 Strips and Tunnels		
3 Pyramids	**X**	
4 Regular Polyhedra		
5 Symmetry (symmetry and pattern)		
6 Colour Patterns (symmetry and pattern & how polyhedra relate in space, in part 9)		**X**
7 Space Fillers (how polyhedra relate in space)		**X**
8 Double edge length tetrahedron (how polyhedra relate in space)		**X**
9 Stella Octangula (how polyhedra relate in space)		**X**
10 Stellated Polyhedra and Duality (how polyhedra relate in space)		**X**
11 Faces and Edges (Euler formula theme)		**X**
12 Angle Deficit (Euler formula theme)		**X**
13 Torus (Euler formula theme)		**X**

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 Start with Pyramids

First the class will carry out the activity for pyramids (figure 1) if not already done.

Figure 1 Pyramids from previous activity

2 Building a tetrahedron

Have the students split into groups of 4 or 5 and have each group make a triangular based pyramid (figure 2). Ask:

How many triangles are there at the vertex on top where all the triangles meet?

3

How many triangles at the other vertices?

2

Now make a closed up shape with 3 triangles at every vertex.

Figure 2 The tetrahedron

How many triangles does your shape have?

4

3 Starting with other pyramids

Again start with a visualisation and prediction task:

Now if you start with a square base pyramid, how many triangles have to be at each vertex to make a closed up shape with the same number of triangles at each vertex?

4

What do you think that shape might look like?

'diamond' or 'rhombus'

How many faces will it have? Each triangle will be a face of the shape.

'6', '4'

If there are sufficient triangles have the each group build with new triangles or if not have them untie their pyramids first.

Go ahead and make a closed up shape with four triangles at every vertex.

Figure 3 The octahedron

Walk around and check that their shapes do in fact have four triangles at EVERY vertex. The octahedron will be the correct result as shown in figure 3. Often there will be vertices with too few triangles. If all groups finish on time, ask the class as a whole, or each groups as they are ready.

How many faces does your shape have?

'6' ,'8'

Bring a shape with 6 and a shape with 8 triangles up to the front. Check you counted the faces correctly.

Yes we counted properly

One group at a time show the class how every vertex has four triangles.

For the shape with six triangles they will find, perhaps with help from the class

This vertex only has three triangles.

Then repeat the above visualization, prediction and building instructions and questions for a pentagonal based pyramid instead of a square base, either on a group by group basis or as a class. The icosahedron will result as shown in figure 4.

Figure 4 The icosahedron

4 Definition of regular polyhedron

The activity now moves to a discussion of the meaning of "regular" for polyhedra. Starting with a vocabulary check, proceed as follows:

What is the difference between a polygon and a polyhedron?

'Flat and solid', 'three dimensional'

What are their definitions?

two dimensional shape with straight sides and a three dimensional shape with flat sides

Refer to your curriculum for these definitions, and definitions of face, edge and vertex. For example

A polygon is a two dimensional shape with vertices connected by straight sides.

A polyhedron is a three dimensional shape with polygonal faces, connected along their edges with no gaps or overlaps.

What is the difference between an edge and a side?

See the discussion in part 8 of Activity **3 Pyramids** for 'edges' versus 'sides'.

Now return to discuss the regular polyhedra (Platonic solids)

If a regular **polygon** must have congruent sides and congruent angles, what must a regular **polyhedron** have?

'All triangular faces', 'same number of faces at each vertex'

The criteria we use are:

- all faces are regular polygons,
- all faces are congruent to each other, and
- the same number of faces meeting at each vertex on the whole shape.

It may be worth going over the vocabulary used in the definition of regular polyhedra. Ask the following questions and correct where necessary.

What does 'all faces are congruent' mean?

'All faces the same'

All the faces have the same number of edges, the same length edges and the same angles.

What does 'regular polygon' mean?

' It is the same all the way round'

What is the same all the way round?

'sides', 'side lengths', 'vertices', 'angles'

Regular polygon means a 2-dimensional shape which is equilateral and equiangular, that means the side lengths are all equal and the angles are all equal.

Although our regular polygons are triangles, you can have regular polyhedra with other faces.

What is another kind of regular polygon?

'Rhombus', 'Square'

Only the square is regular because a rhombus can have different angles at its vertices. What polyhedron has square faces?

Cube

5 Etymology of names of regular polyhedra (Platonic solids)

Now proceed to naming the regular polyhedra (Platonic solids).

If a fourth triangle is attached to the base of the triangular pyramid then there will be 3 faces at each of its vertices, so it is a regular polyhedron, named a *tetrahedron* (tetra is a Greek prefix for 4; quad is the Latin prefix for 4). This is just like the game, Tetris, uses arrangements of 4 squares that drop down to build the wall in the game.

We sometimes call regular polyhedra 'Platonic solids' after the Greek mathematician and philosopher, Plato.

6 Tabulating numbers of faces, edges and vertices

Set up a table on chart paper or on the board with these headings. As participants determine the number of faces on each figure, write its name under the generic heading. Leave space between each figure's data as shown below as these will be filled in later for figures with intermediate numbers of faces.

Determining the number of faces on the icosahedron (icosi is a Greek prefix for 20) is challenging and should not be told to participants. Initially it is difficult to determine if the figure is lying on one of its faces. But when lifted onto a vertex as shown in figure 4 above, the rotational symmetry helps.

There is a pentagonal pyramid pointing up and another pointing down, accounting for 10 faces. Then, there are 5 triangles hanging down from the upper pyramid into the zig-zag belt, and 5 pointing upwards from the lower, upside-down pyramid, for a total of 20 triangles.

Polyhedron	**Faces**	**Edges**	**Vertices**
Tetrahedron	4		
Octahedron	8		
Icosahedron	20		

Participants now determine the numbers of edges and vertices on each figure. The icosahedron may again be difficult, especially for finding the number of edges. Again orienting the figure on one of its vertices (figure 4) so that the upper and lower pentagonal pyramids can be visualized easily is helpful. Another method consists of the following: each student grabs hold of one distinct edge in each hand, enough so that all edges are held, then students call out successive numbers to count both the edges they are holding, until all are called out and the total is reached.

7 Relating faces and edges

As this part of the activity overlaps with part of the Faces and Edges activity it could be postponed until then. Otherwise continue as follows.

After students have determined the number of edges of the icosahedron by direct counting, ask participants to compute the number of edges of the icosahedron by using the number of triangular faces as follows:

20 triangles X 3 edges per triangle = 60 edges; but since the edges are tied together in pairs, we divide by 2, giving 30 edges

Try this method for the tetrahedron and the octahedron and check if it gives the correct numbers of edges.

Note that because all faces are triangles, this relates to a ratio in the table between the column of Faces and the column of edges.

8 The correct table entries

Polyhedron	Faces	Edges	Vertices
Tetrahedron	4	6	4
Octahedron	8	12	6
Icosahedron	20	30	12

9 More than 5 triangles at a point?

We have constructed all possible regular polyhedra from equilateral triangular faces, using 3, 4, and 5 per vertex.

Try to connect 6 equilateral triangles together at every vertex to make a closed up shape. What happens?

It won't close up, it just goes flat and keeps going.

Can you try to add in extra triangles to form 7 or 8 faces at the central vertex, what happens?

It goes floppy.

So only 3, 4, and 5 triangles at a vertex give us a closed up shape.

10 Other regular polygons: squares

Since we have used equilateral triangles for these regular polyhedra, maybe we can use a different regular polygon and make some more regular polyhedra.

What is the next regular polygon we can try?

Square

Proceed in exactly the same way, asking:

how many squares can be placed around a single point.

4

We have used Polydron™ shapes to demonstrate this part of the activity since the polygons (triangles, squares, pentagons, hexagons) all have the same side lengths and easily snap together. Otherwise, use laminated polygons that can be taped together or cut apart as needed.

Figure 5 Four and three squares around a point

Four squares will completely surround a point and leave no gaps and will not overlap. Now carry out a visualisation and prediction activity.

If you remove one square and connect the two free edges, what will happen?

Makes a corner

What kind of corner?

Like the corner of a box

Go ahead and close up the edges.

Figure 6 A polyhedral vertex with 3 squares

The figure pops up, like the tetrahedron to form three faces of a cube, meeting at a single vertex.

What will it be when it is completed so every vertex has 3 squares?

Cube

Try it and complete the shape

What can we call it?

Cube

How many faces does it have?

6

So what other name could we have for it?

A hint can be offered

What is the name of a polygon with six edges?

Hexagon

So a polyhedron with six faces is a …?

Hexahedron

11 Verify the cube is a regular polyhedron

This reviews the definitions and vocabulary

Is this a regular polyhedron?

Yes

Are all faces congruent?

Yes, they are all squares

Are there the same number of faces at each vertex?

Yes, 3

Fill in the numbers of faces, edges and vertices in the table.

Polyhedron	**Faces**	**Edges**	**Vertices**
Tetrahedron	4	6	4
Hexahedron (cube)	6	12	8
Octahedron	8	12	6
Icosahedron	20	30	12

We can construct a polyhedron with only 3 squares at each vertex (2 forms a flat, folded shape, and 4 at each vertex forms a flat surface).

12 Other regular polygons: pentagons

In class ask

How many pentagons do you think will fit around a point?

Pass out pentagons for students to try and ask either in class or in groups

What did you find?

3

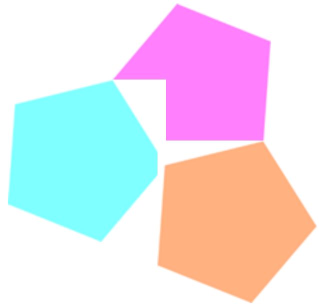

Figure 6 Three regular Pentagons meet at a point

Does it close up flat around the central vertex, or can you get a fourth one in? What happens if you try?

'It won't go', 'the gap is too small'

If we try to insert a fourth, then the figure ruffles, like when we tried to connect 7 or 8 triangles at a vertex.

Ask a participant or the groups to complete the figure made from pentagons.

13 Counting the faces edges and vertices on a dodecahedron

Determine the number of faces, edges and vertices.

Next, say:

dodeca means 12, so what do we call a polyhedron with 12 faces?

Dodecahedron

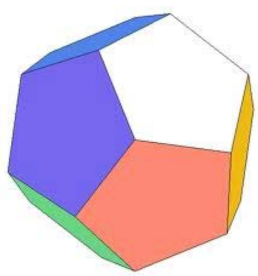

Figure 7 The dodecahedron

To confirm the number of faces proceed as follows, demonstrating as you explain:

The number of edges can be determined by orienting the figure with one pair of faces parallel to the floor.

How many faces are hanging down from the top face,

5

How many faces point up from the bottom face,

5

So how many faces altogether in the polyhedron

12 faces.

Now the power of using ratios to help counting can be used to good effect

How many edges does a face have?

5

So for 12 pentagons, how many separate edges in total?

12 x 5 = 60

How do we get the total number edges on the polyhedron?

divide by 2

Why?

The edges are tied in pairs

What does that give?

30

Similarly, each face contributes 5 vertices, each being shared by 3, producing (5 X 12)/3 = 20 vertices:

How many vertices does a face have?

5

So for 12 pentagons, how many vertices in total?

12 x 5 = 60

How many separate pentagon vertices meet a dodecahedron vertex?

3

So what do we have to do now?

Divide by 3

The vertices come together in threes

What does that give?

20

With one face flat on the floor can you check the number of vertices?

They come in four horizontal rings of 5 vertices. This is useful to point out in relation to rotational symmetry.

Polyhedron	**Faces**	**Edges**	**Vertices**
Tetrahedron	4	6	4
Hexahedron (cube)	6	12	8
Octahedron	8	12	6
Dodecahedron	12	30	20
Icosahedron	20	30	12

14 Any more polygons we can use?

How do we know we found all the Platonic solids?

We tried all the shapes

We used triangles, squares and pentagons, is there anything else we can use?

What happens if we use hexagons?

Let them try assembling hexagons

It just goes flat

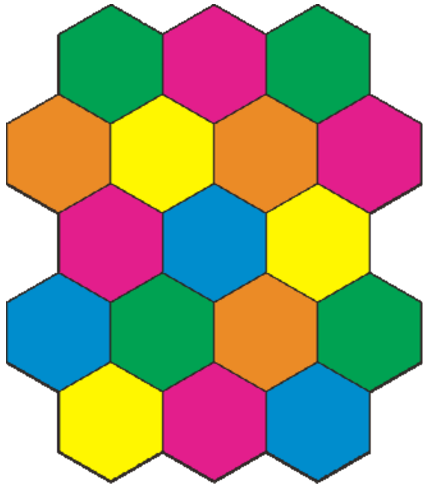

Figure 8 Hexagons fill the plane

We have shown that a regular hexagon (like the one formed from 6 equilateral triangles) will completely fill the flat space.

And, if we try to use regular heptagons 7-sided polygons, what happens?

only 2 can be connected and therefore fold flat together like a sandwich

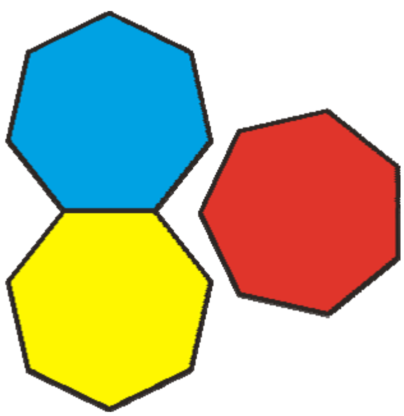

Figure 9 Three Heptagons do not fit together

Show the pictures of the hexagonal tiling and heptagons fitting together.

Have we tried all the shapes now? What about octagons?

You can't fit 3 together

What about more than 8?

Won't fit

You can go on to discuss or explore why this is true. As the number of sides of the polygons increase, the gap, (in the figure 9 between the blue and yellow polygons) will get smaller, and the angle at a vertex of the third polygon that might fit into the gap (the red one in figure 9) will get larger. So the third polygon will never fit for any number of vertices above 6. In terms of angle measure, 120 degrees is the maximum angle that will go into 360 three times. This is the angle at a vertex of a regular hexagon.

Now summarize:

Thus, we have constructed the only 5 regular polyhedrons known to man (by the Greeks in Plato's age). These are called the Platonic Solids.

15 Patterns in the table

Polyhedron	Faces	Edges	Vertices
Tetrahedron	4	6	4
Hexahedron (cube)	6	12	8
Octahedron	8	12	6
Dodecahedron	12	30	20
Icosahedron	20	30	12

Now we know we have all our entries filled in, what patterns can you see in the table?

'The numbers are all even.'
'There are always more edges than faces or vertices'

'There are 2 figures with 12 edges and 2 figures with 30 edges'

What do you see about the numbers for the cube and the octahedron?

The numbers of faces and vertices switch

Does that happen anywhere else?

for the dodecahedron and the icosahedron

They are called, *duals* of each other. The tetrahedron is its own dual since it has 4 faces and 4 vertices.

Finally you can mention Euler's Formula

The number of edges is always 2 less than the sum of the faces and vertices in each figure; i.e.:

$$E = F + V - 2$$

This is known as Euler's Formula. Question: I wonder if this is true for other polyhedrons that are not