5 Symmetry

Themes	Reflection and rotational symmetry in three dimensions.
Vocabulary	Line of reflection, plane of reflection, centre of rotation, axis of rotation, perpendicular.
Synopsis	Review 2 dimensional symmetry. Identify and mark with tape all reflection planes on a tetrahedron. Find rotational symmetry axes using the turn and stop game. Investigate compositions of reflections and rotations. Investigate other polyhedra.

Overall structure	**Previous**	**Extension**
1 Use, Safety and the Rhombus	**X**	
2 Strips and Tunnels		
3 Pyramids	**X**	
4 Regular Polyhedra		
5 Symmetry		
6 Colour Patterns (shows more symmetries for the octahedron and relates symmetry to colouring patterns)		**X**
7 Space Fillers		
8 Double edge length tetrahedron		
9 Stella Octangula		
10 Stellated Polyhedra and Duality (examines symmetry in the contest of dual polyhedra)		**X**
11 Faces and Edges		
12 Angle Deficit		
13 Torus (illustrates some advanced symmetries)		**X**

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 Review symmetries of the triangle and the rhombus

Review the 3 lines of reflection and 1/3 turn rotational symmetry of the triangle, and the 2 lines of reflection and ½ turn rotational symmetry of the rhombus. Emphasize that terms 'line of reflection' and 'centre of rotation' only apply to symmetries of 2 dimensional figures. See also activity **1 Introduction**, for discussion of these symmetries.

2 Identify one or more reflections

We start by identifying reflective symmetry in a tetrahedron. Placing the shape in particular orientations and viewing from particular angles can help make these apparent such as the vertical plane of symmetry in figure 1.

Figure 1 Viewing a vertical plane of symmetry

Can anyone see a reflection symmetry in the shape?

Down the middle

Show the class which way with your hands.
If it was a mirror, show where the mirror would go.

Using masking tape, mark where the plane of reflection goes through as shown in figure 2.

Note: The following images are all seen from above.

Figure 2 Taping to mark a plane of reflection

The reflection plane cuts through these two faces where the tape is, and where else does it cut through?

The middle, the bottom edge

Turn the shape to continue the taping along the bottom edge as in figure 3.

So altogether the plane goes through two faces and one edge. If we tape along the edge, the tape now joins up with itself after going all the way around.

Figure 3 Tape following the plane of symmetry all the way round the shape.

Other planes of reflection can also be taped on interactively with the class or in groups, as shown in figure 4 below. Note that the pattern made by the tape looks the same on all faces.

3 Finding rotations: The turn and stop game

Proceed as follows to use the turn and stop game to identify rotational symmetry. We use a 1/3 turn rotational symmetry on the tetrahedron as an example. Figure 4 shows the starting position with the tetrahedron resting on the ground or floor. Tell the class:

> Everyone gather round the tetrahedron but far enough back so everyone can see.

Start drawing their attention to specific features of the tetrahedron and their orientation:

> Which way is the yellow face pointing?

'Toward the window',
'the wall'

Figure 4 The starting position

Say:

> Look carefully at the position of the shape which way are the vertices or edges or faces pointing or facing as you are looking at them?
>
> Now stay still while I rotate the shape and when there are faces edges and vertices in the same places in the same direction say 'Stop!"
>
> We will count how many times we stop as we go until we get back to where we started on the last stop.

Start turning the shape:

Stop!

Figure 5 Stop No 1

Ok that is ONE position with faces, edges and vertices in the same places, now we will keep going.

Start turning again:

Stop!

That is TWO.

Figure 6 Stop No 2

Start turning again:

Stop!

That is THREE, and are we back where we started yet?

Yes

Figure 7 Stop No 3

Which way is the yellow face pointing?

The same way as when we started.

So how many positions altogether?

Three

We call the line everything rotated around the axis of rotation. Here it is a vertical line going up from the page.

We stopped in three different positions in the whole turn once around until we were back to where we started. So what fraction of a turn do we do each time?

1/3

Also the amount we turned each time, the angle we turned through was the same. How many degrees was each turn between stops?

120

If your curriculum uses the term ‘order 3 rotation’, you can discuss this here. Mention that there are 3 positions in the whole turn where the shape looks the same; if the 1/3 turn is done 3 times you return to the original position; features come in groups of 3 that are interchanged by the 1/3 turn.

Repeat the exercise for half turn rotational symmetry by holding the tetrahedron on the mid point of an edge, letting it hang down under its own weight.

Note axes of rotational symmetry for a 1/3 turn will pass through a vertex and the midpoint of the opposite face, and for a 1/2 turn, the mid points of a pair of opposite edges.

At this point, it is possible to allow the students to break into groups and build other shapes with 6 or 8 triangles and to look for reflections and rotations in those shapes. Say:

> In groups of 4 or 5 students take 6 or 8 triangles and make another shape and identify its symmetries.

4 Combining two reflections

Give the instructions to the students in groups.

> Tape around two planes of reflection on one of your shapes that meet at 90 degrees to each other. That is two perpendicular reflection planes.

Go round the groups to ensure they are picking perpendicular planes.

> What happens if you do one reflection followed by the other.
>
> You can consider what happens to the midpoint of a diagonal edge when each of the reflections is performed one after the other.

Take hold of one of the laces on a diagonal edge to show what you mean by edge midpoint.

Figure 8 Two reflection planes intersect on an edge.

Can anyone show where this edge midpoint goes after the first and then the second reflection?

Touch each position with your hand so everyone can see.

Does it make any difference which order you do the reflections in?

No

What is the transformation that results from combining the two reflections

Rotation

How many degrees?

180

About what axis?

'Where the planes meet'
'the vertical centre'

Yes. The axis is where the planes intersect.

Is this also an axis of rotational symmetry for the shape?

Yes

What kind?

½ turn

Now try for two planes of reflection which are not perpendicular to each other.

If students need help finding reflection planes that are not perpendicular, place a tetrahedron with all reflection planes marked in tape as in figure 4, on the floor on a face. There are three vertical planes indicated by tape, none of which are mutually perpendicular.

Do you still get a rotation when you combine the reflections?

Yes

What angle?

120 degrees

Does the order of carrying out the reflections make a difference?

Yes, changes the direction of rotation.

Where is the axis of rotation in relation to the reflection planes?

Up through the middle, where they cross each other.

Yes the axis of rotation is the vertical line where the reflection planes intersect.

When the reflection planes were at a 90 degree angle, what was the angle of rotation?

180 degrees

What is the angle between the planes now?

60 degrees

What is the angle of rotation?

120 degrees

So what seems to be the rule here?

The rotation angle is twice the angle between the reflection planes

For advanced students, this can be related to a proof in two dimensions that the combination of two reflections in lines that are not parallel gives a rotation through twice the angle between the lines, and that the order in which reflections are carried out determines which way the rotation turns the shape, clockwise or counter clockwise.

5 Find and tape all reflections and rotations

If the groups have not already done this earlier, ask the groups:

Tape around all the planes of reflective symmetry all the way round the face.

What patterns do you notice?

Is there a connection between the symmetries of the equilateral triangle and the symmetries of the tetrahedron?

They are the same

Yes almost. We can say the symmetries of the triangular face extend to symmetries of the whole shape. So a line of reflection on the face is part of a plane of reflection on the whole shape. A centre of rotation on the face is part of an axis of rotation on the whole shape.

Figure 9 Taping round all reflection planes on a tetrahedron (the middle image is not completely taped)

6 Counting the planes of reflection

How many planes of reflection can you count?

3, 4

Note each plane of reflection goes completely along one and only edge. Each edge has a plane going through its whole length.

How many edges are there?

6

So how many planes of reflection

6

7 Counting the axes of rotation

How can we count the axes of rotational symmetry?

There is one for each vertex and one for each edge

What kind of rotational symmetry has an axis going through a vertex?

1/3 turn

What kind of rotational symmetry has an axis going through an edge?

½ turn

Instruct the student who answers:

HOLD IT UP TO DEMONSTRATE that to the rest of the class.

Now continue the discussion:

How many vertices are there?

4

Does everyone agree there are four axis of 1/3 turn rotations, one for each vertex?

Yes

Someone come and DEMONSTRATE the four axes for 1/3 turn rotational symmetry to the rest of the class.

How many edges are there?

6

Does everyone agree there are 6 axes of ½ turn rotational symmetry, one for each edge?

No, you are counting twice

Why

Each axis goes through two edges

Tell a student who answers, or one who has not demonstrated before:

HOLD IT UP TO DEMONSTRATE that to the rest of the class.

So what do we have to do to get the right number of axes?

Divide by 2

Does everyone agree there are 3 axes of ½ turn rotational symmetry one for every opposite pair of edges?

Yes

And in total how many axes of rotation?

7

Yes three for ½ turn and four for 1/3 turn rotational symmetries.

8 Combinations that are neither reflections nor rotations. (Advanced)

If a cube or octahedron is reflected in three planes that are perpendicular to each other, then the result is neither a rotation nor a reflection.

What is it?

For a hint, consider this in X-Y-Z coordinates, and reflecting in the three planes, X=0, Y=0, Z=0.

The result is an inversion, each point is taken to the opposite point on the cube, such that the original point and its image are at equal distances from the centre of the cube in opposite directions along a straight line. Also see the extension activity: **13 The Torus.**

