8 Double edge length tetrahedron

Themes	Space filling with polyhedra, number patterns length area and volume scaling.
Vocabulary	Length, area and volume scaling.
Synopsis	Construct a net for a double edge length tetrahedron, noting that the sum of the first n odd numbers is n squared. Fill the double edge length tetrahedron as far possible with single edge length tetrahedra and add a ‘mystery shape’, the octahedron. Using scaling and volume subtraction deduce the 4 to 1 ratio of volumes of a regular octahedron to a regular tetrahedron of equal edge length.

Overall structure	**Previous**	**Extension**
1 Use, Safety and the Rhombus	**X**	
2 Strips and Tunnels		
3 Pyramids	**X**	
4 Regular Polyhedra	**X**	
5 Symmetry		
6 Colour Patterns		
7 Space Fillers		
8 Double edge length tetrahedron		
9 Stella Octangula		**X**
10 Stellated Polyhedra and Duality		
11 Faces and Edges		
12 Angle Deficit		
13 Torus		

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 A net for a single edge length tetrahedron

Lay out four triangles as shown in figure 1 and have students tie the edges together.

Figure 1 Net of a unit edge length tetrahedron

Now ask:

How can we make this into a solid shape?

Lift the corners up.

Show us.

Once everyone has seen how this makes a tetrahedron, ask:

This is an edge length 1 tetrahedron, a unit edge length tetrahedron.

How could we make a double edge length tetrahedron?

Start with a bigger triangle

How many triangles do we need?

'4', '8', 'try it and see'

Let's make it bigger and see what works. We will make another large triangle with one more row of triangles. What do you think will happen?

'It will make a bigger tetrahedron'
'It won't work'

Make the edge length 3 triangle by adding another row.

Figure 2 The edge length three triangle

Try to fold it up into a tetrahedron

You can't do it

How can we make it work?

Make it bigger

Add another row of triangles make the edge length 4 triangle.

Figure 3 The edge length four triangle

Do you think it will close up?

'Maybe', 'yes'

Lift and hold it in position to see if it works.

Yes it works

2 Number patterns I: Square numbers

Let's lay all the triangles out flat again and count how many triangles we used each time.

For each edge length how many triangles did we need ?

You may want to redraw the triangles on the board as in figures 1, 2 and 3, or make separate triangles of edge length 2, 3 and 4.

Sketch and label the triangles of edge length 1, edge length 2, edge length 3 and edge length 4.

Put the following table on the board or hand out copies

Edge length	**1**	**2**	**3**	**4**
Number of single triangles	1			

Copy the following table and fill in the numbers. The first column is just 1 triangle and so has edge length 1

Can anyone see a pattern in the numbers?

The correctly completed table is as follows:

Edge length	**1**	**2**	**3**	**4**
Number of single triangles	1	4	9	16

'The bottom is the top times itself' 'The bottom is all square numbers'

3 Number patterns II: Sums of odd numbers

Carry out the following dialogue, writing results on the board as shown beneath. You may want to point to the triangles of draw on the board, whichever works best.

Let's look at how many single triangles are in the rows
of these larger triangles

Pull out or draw the triangles in figure 4

Figure 4 The edge length 2 triangle

How many triangles in this first row?

Point to a single triangle at an apex and say:

This is the first row.
How many triangles does it contain?

One triangle

And in the second row

Three triangles

How many altogether?

4

Yes, I am going to write on the board 1+3=4

and continue with the two other triangles until you get to 16, writing results on board to give:

1+3=4,

1+3+5=9,

1+3+5+7=16,

Now ask:

Can you see any number patterns?

'All odd numbers on the left'. 'Square numbers on the right'

How many odd numbers in the first line?

2

And in the second line

3

and in the third

4

Can anyone see a connection between the number of odd numbers on the left and the number on the right?

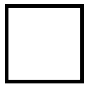
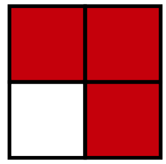
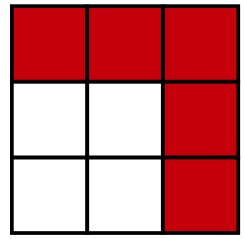
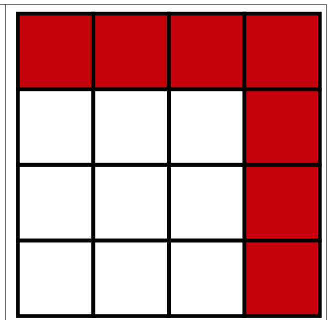
It is squared

The pattern that the sum of the first n odd numbers is n squared, has now been shown for n up to 4.

Using language the students will understand, such as the following for students not familiar with variables, summarize what has been found in a sentence. You can also write the statement on the board.

Yes the sum of a sequence of odd numbers going up in order 1,3,5 etc, starting at 1, is the number of odd numbers times itself.

The following graphic, gives some insight into this pattern. You can use it to relate this number pattern to squares as well as to triangles.

			
1	1+3=4	1+3+5=9	1+3+5+7=16

4 Discuss length and area scaling

We found that the number of single triangles is the square of the edge length. If the area of one triangle is one unit area, what is the area of the edge length 2 triangle?

4 units

Why?

because it is made up of four triangles

We can talk about scale factor. The edge length 2 triangle has a length scale factor of 2 compared to the edge length 1 triangle, because the length has doubled. The area is four times as much on the edge length 2 triangle compared to the edge length 1, so the area scale factor is 4.

We can fill in the table with the same numbers as for edge length and number of triangles.

Edge length scale factor	**1**	**2**	**3**	**4**
Area scale factor	1	4	9	16

So, if you double the edge length what happens to the area?

It goes up by 4

Is that add 4 or multiply by 4?

Multiply

If the scale factor for length is 2 the scale factor for area is 4.

If you triple the edge length what happens to the area?

Multiply by 9

If the scale factor for length is 3 the scale factor for area is what?

9

If you multiply the edge length by 4 the edge length what is the scale factor for area?

16

Is there a pattern?

The scale factor for area is the square of the scale factor length

How do we measure length, what units do we use

Centimetres

What about area

Centimetres squared

What about volume?

Centimetres cubed

The scale factor for volume is the cube of the scale factor for length. So we can write, using '*n*' for the scale factor of length, when all lengths are multiplied by '*n*' in scaling up a figure changing only its size and keeping its shapes and angles the same.

Write on the board and say:

length, cm, length scale factor = n

area, cm^2, area scale factor = n^2

volume, cm^3 , volume scale factor = n^3

5 Constructing the double edge length tetrahedron

Reassemble the double edge length tetrahedron again and tie the edges together.

Figure 5 The double edge length tetrahedron

For now, let's call the area of a single triangle a unit area.

How many units of area is each large face of this large tetrahedron.

4 units

How many large faces

4

Total area in units of the large tetrahedron is?

16

16 units

What is the area of a single edge length tetrahedron?

4 units

Why

It has 4 triangles

In going from a single edge length tetrahedron to a double edge length tetrahedron, that is a length scale factor of 2.

What is the area scale factor to go from an area of 4 to an area of 16?

4

Is the area scale the length scale factor squared?

Yes

16 is 4 squared

6 Filling the double edge length tetrahedron

Partly untie the double edge length tetrahedron to enable a window to be opened and let the students all look inside at the interior space. See figure 6.

Now have the students construct a number of single edge length tetrahedra and once they are built, ask them to estimate:

> How many single edge length tetrahedra will fill the double edge length tetrahedron?
>
> *'5', '6', '7'*

Do not expect students to use volume scaling to calculate 8 tetrahedra, but this volume scaling will be used later. Do not hint at it yet!

Figure 6 Open window

Next untie enough knots to enable the single edge length tetrahedra to be inserted, as in figure 7.

Figure 7 Inserting a tetrahedron

> Now see how many single edge length tetrahedra you can fit in to fill the double edge length tetrahedron.

Once the students get to 5 or 6, they will find they can't fill it without gaps. Once they have been left to try to work with this for a while, remove all except three tetrahedra in the bottom corners, as in figure 8. Ask students:

look inside and describe the gap that is created.

If the fourth single edge length tetrahedron is added in the top, how many faces will the gap have?

You may need to temporarily re-insert the fourth single edge length tetrahedron to show them the shape of the gap in the middle

Figure 8 The space for the mystery shape

What shape is the bottom face of the space?

A triangle

How many faces around the sides?

6 triangles

Which way are the side triangles pointing? Up or Down?

Both

How many pointing up and how many pointing down?

3 each

That is three pointing up which are touching the tetrahedra we put in the corners and three others pointing down.

Reach in and touch the triangles.

How many triangles on top of the gap touching the top tetrahedron.

1

How many triangles altogether will make the shape in the gap?

8

Take 8 triangles and try building it, and see if it fits.

As they work it may be worth drawing their attention to how many triangles there are at each vertex in the gap. In which case ask while reaching inside:

How many triangles come together at each vertex in the gap?

4

The shape is a regular octahedron and the students will fit it in once it is built. This construction from observation of aspects of the faces of the gap will help them develop their geometric thinking. Ask them

What is the mystery shape you just made that fits the space and are there any gaps now?

Octahedron, no more gaps

Finally remove all the shapes and have them construct a double edge length tetrahedron by stacking the four tetrahedra with the octahedron as in figure 9 and place the two double edge length tetrahedra side by side.

Figure 9 The two tetrahedra side by side

7 The relative volumes of the octahedron and tetrahedron

What is the scale factor for length from a single edge length tetrahedron to a double edge length tetrahedron.

2

How do we get the scale factor for surface area?

Square it

So what will that be

4

How do we get the scale factor for volume?

Cube it

Which gives what as the volume scale factor?

8

Yes, 2 cubed = 8

Go over to the shapes, pointing to the first hollow double edge length tetrahedron, and say

Now from scaling we know the volume of one of these double edge length tetrahedron is 8 times the volume of a single edge length tetrahedron.

Now go to the stacked tetrahedra and octahedra and say:

But we also know that it is equal to the volume of these 4 single edge length tetrahedra plus the volume of this octahedron.

What does this tell us?

The octahedron equals 4 tetrahedra

The volume of four single edge length tetrahedra is equal to the volume of a single edge length octahedron.

