Proof of the angle deficit theorem for a polyhedron with triangular faces

The angle deficit theorem relates the geometric quantity of angle measure over a polyhedron to the numbers of faces, edges and vertices of the polyhedron. This perhaps surprising connection between angle and number can be seen as relating to the fact that the angles of a triangle always add up to 180° (π Radians). Two versions of the proof are given. One version of the theorem is expressed in degrees, and the other in Radians. The quantities, 360° and 2π Radians, as they appear in the two versions of theorems, account for whether the unit of measure is degrees or Radians.

In both proofs we use the following symbolic notation:

- F is the number of triangular faces.
- E is the number of edges.
- V is the number of vertices.

Proof that total angle deficit in degrees = 360°×(F–E+V)

Total angle deficit = Sum of angle deficits at each vertex

Total angle deficit = Number of vertices×360°–sum of all angles at all vertices

= V×360°–sum of all angles in all triangles

Using the fact that the angles of a triangle add up to 180°, we can write

Sum of all angles in all triangles = 180°×number of triangles

= 180°×F

= 360°×V–180°×F

If E=3F/2, then 3F=2E, as shown in the faces and edges activity, therefore:

$$V-E+F = V-3F/2+F$$

which simplifies to:

$$V-E+F= V-F/2$$

And finally multiply both sides by 360°

$$360°\times(V-E+F) = 360°\times V-180°\times F = \text{Total angle deficit}$$

Proof that total angles deficit in Radians = 2π×(F–E+V)

Total angle deficit = sum of angle deficits at each vertex

Total angle deficit = number of vertices×2π−sum of all angles at all vertices

$$= V\times 2\pi - \text{sum of all angles in all triangles}$$

Using the fact that the angles of a triangle add up to π, we can write

$$\text{Sum of all angles in all triangles} = F\times\pi$$

$$\text{Total angle deficit} = V\times 2\pi - F\times\pi$$

If $E = 3F/2$ so $3F = 2E$, as shown in the faces and edges activity, therefore:

$$V-E+F = V-3F/2+F, \text{ which simplifies to}$$

$$V-E+F = V-F/2$$

And finally multiply both sides by 2π

$$2\pi\times(V-E+F) = 2\pi\times V-\pi\times F = \text{Total angle deficit}$$

Other polyhedra and discussion

To consider a polyhedron with polygonal faces other than triangles, consider subdividing each n-gon into triangles. You will see that the values of F−E+V remains the same, before and after subdivision because each time you add a face you also add an edge. Also, the amount of angle at a vertex remains the same before and after subdivision. Therefore the formula will be true for all polyhedra with any polygons as faces.

The quantity F−E+V is called the Euler characteristic. For convex polyhedra and polyhedra with no tunnels the Euler characteristic is 2.

In this special case, sometimes the term Euler formula is used for the equation:

$$F-E+V = 2$$

However, if there is a tunnel, such as in the torus, the Euler characteristic is 0. If we call the number of tunnels the genus, g, then we have the more general formula:

$$F-E+V = 2-2g$$

So a torus has genus 1, and $F-E+V = 0$

It is also valid to think angle deficit at a vertex as a kind or curvature. This is known as Gaussian curvature and is explained in the reading below.

For further reading see:

Alexandrov, A.D. & Zagaller V.A., (1976). Intrinsic Geometry of Surfaces, Translations of Mathematical Monographs vol. 15, American Mathematical Society, Providence, Rhodes Island, 1967. Page 8.

Do Carmo, M.P., Differential Geometry of Curves and Surfaces, Prentice-Hall Inc., Englewood-Cliffs, New Jersey. Page 274.