The Aesthetics of Colour in Mathematical Diagramming

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Abstract

Mathematical diagramming serves many purposes in mathematics research and education. It may be conceptual or representational in form, but in all cases, producing graphic presentations benefits from carefully considered use of the various elements of graphical composition. The challenge of mathematical diagramming – indeed, of all diagramming – is how to do it well, so that the diagram fulfills its specific conceptual and communicative purposes. This paper and workshop will examine how the considered use of colour in diagramming and in corresponding artefacts can identify and clarify mathematically pertinent features with respect to various mathematical insights. The idea and role of aesthetics in diagramming will be considered with respect to various representations and artefacts of interlacing and some of the related sensitivities, especially colour sensitivity, will be exercised. The reader should note that since the proceedings are printed in black and white, it is worth consulting the electronic version in the Bridges Archive[1] to see the colour images.

Introduction

What are the criteria for a good mathematical diagram? General aims for any good diagram are clarity, understanding, and engagement, in both mathematical and graphical roles. A good mathematical diagram compels you to ask mathematical questions and helps you to arrive at mathematical insights. That is achieved in part by the suitability and aesthetic qualities of the graphical decisions made in the process of diagramming. This process, distinct from any end product, has its own problematic; it may offer its own insights.

As a collective we are particularly interested in the mathematical elements in making [1]: the act of making provides its own unique insights. This currently involves interrogating assumptions and deconstructing both diagrams and the process of diagramming. Questions raised include: How do diagrams come into being? How do people learn to make, use and read diagrams effectively? How does this relate to mathematical intentions? What are some challenges specific to mathematical diagramming? How can visual aesthetics inform diagramming? These questions are generated through the multiple

[1] http://archive.bridgesmathart.org/

perspectives provided by the group's diversity. Similarly, we expect that the processes that will be highlighted in this workshop will provide multiple entry points and foci for discussion.

In this paper, we focus on colour as one of the most vivid and engaging graphical elements. Colour can be used as a tool for navigation, communication, and pedagogy in diagrams. In mathematical diagrams, it can be deliberately used to identify and distinguish particular variables of a situation, to highlight or trace similarities or relationships, and thereby to clearly depict and communicate characteristics of a problem or situation that might be difficult to explain verbally or with symbols alone.

Aesthetic character is a pertinent consideration for any graphic image that aims to communicate meaning. Here, we use the term *aesthetic* in its wider, pre-Baumgarten [2], value-free denotation of "apprehension through the senses", not its modern alternative connotation of "subjective value judgment", i.e. it looks pretty. This denotation focuses on meaningful cognition, as opposed to sensory pleasure. We might term this a *meta-aesthetic* role, in that it serves a specifically conceptual, cognitive purpose, apart from a mere pleasurable one. Such a meta-aesthetic notion depends on the development of *sensitivities*, in both the realms of graphic arts and mathematics. These sensitivities to various graphic elements are important in the apprehension and creation of engaging and effective graphic communication because they guide the eye through and around the depicted forms, stimulating understanding. Sensitivity to form also emerges through the manner in which the colouring is done. Perhaps the most challenging sensitivities involve the appropriate tailoring of the various graphic components to suit the ideas being represented so that they are effectively communicated.

Analogously, we define *mathematical* sensitivities in terms of the competent apprehension and perception of the intrinsic mathematical features and structures of observable artefacts, processes and phenomena, including graphic forms such as diagrams. These sensitivities establish the mathematical foundations of skills, concepts, understanding, proficiency and creativity [3].

Both graphic and mathematical sensitivities come together in the creation of a good mathematical diagram. The effectiveness and engagement of the diagram is determined in part by the particular purpose or intention of the diagram in the context – what it is designed to show, say, or accomplish – and in part by the aptness, clarity, and vivacity of the formal composition. It is in the skillful and insightful fitting together of all these various elements that diagramming can become an art form, integrating visuo-aesthetic satisfaction with a potent pedagogical or thinking tool.

Of particular interest here is the manner in which shading and colouring can affect sensitivity to form. Careful, purposeful use of colour in diagrams can highlight mathematically cogent patterns and concepts; it can be used not only to identify but also to *discover* mathematical concepts and relationships. There is an intrinsic logic in the deliberate use of colour in such situations, whose primary aim is to communicate particular information to the viewer.

A map provides a good example of such use of colour. The colourings on a map serve to highlight various types of continuity, and at the same time to identify distinctions, relative to a convention or stipulated colour key.

A further, well-considered example of this idea can be found in Oliver Byrne's masterpiece: "The First Six Books of The Elements of Euclid, in which Coloured Diagrams and Symbols are used instead of Letters for the Greater Ease of Learners" [4]. The diagrams of Euclid's proofs in this book use colour to signify the various elements cited in each proof, instead of the more traditional letters. In this case, colours replace letters as identifiers, both changing the overall aesthetic effect and, of course, necessitating a different accompanying text. They are, however, used mainly to serve the same purpose as the letters, that is, to mark or highlight the individual, discrete yet salient elements of the diagram, as Roman or Greek alphabet letters do in more standards diagrams.

Colouring interlacing knots and weave diagrams

The use of colour can help depict a different kind of identity and relationship among and between elements of a diagram: it can not only communicate the identity of individual elements but also dis-/connectivity and adjacency between them. This paper will examine the use of colour as an important meta-aesthetic element of visual composition that can communicate identity, connectivity, adjacency and more. To illustrate this we consider, in parallel, depictions of *interlacing knots*[2] as a traditional design, and the diagrams our group uses in our work investigating paper-strip weaving (see [6, 7 and 8]).

Interlacing knots are more often drawn as a decorative art form rather than as an instructional map of something intended to be physically made. Such knot designs, with their over-under stylization, usually drawn in black and white, depict a structure that is analogous to our paper-strip weavings. The meta-aesthetic use of colour we discuss here serves to highlight this parallel.

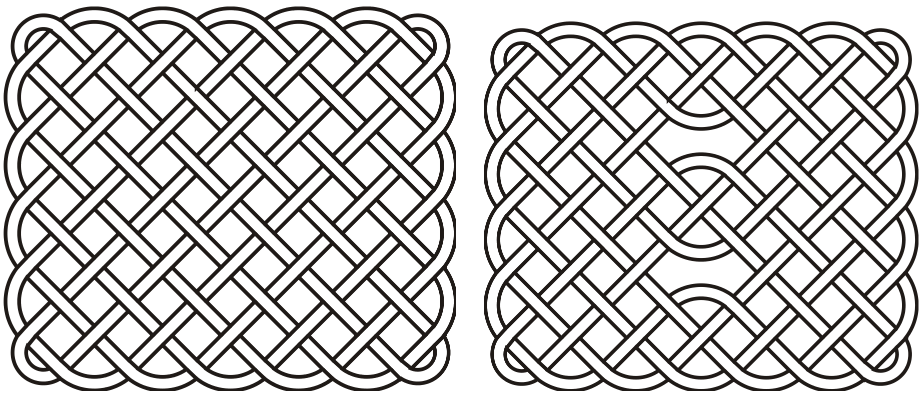

Figure 1: *Example of interlacing knot designs/diagrams*

It might appear that typical interlacing knot drawings such as those in Figure 1 illustrate a lack of colour, using only lines. If the space between the paired lines is interpreted as a ribbon or other element of substantial width, however, it can be given a colour that is distinct from its background (see Figure 2). This makes it clear that what is depicted are continuous, circuitous *paths* taken by discrete elements [6], [9]. This continuity is indeed the attraction of interlacing knots as well as the key challenge in their creation. The black lines illustrate the *edges* of each element as it passes over and under itself and others. Using colour highlights both the identity of the elements and the continuity of their paths.

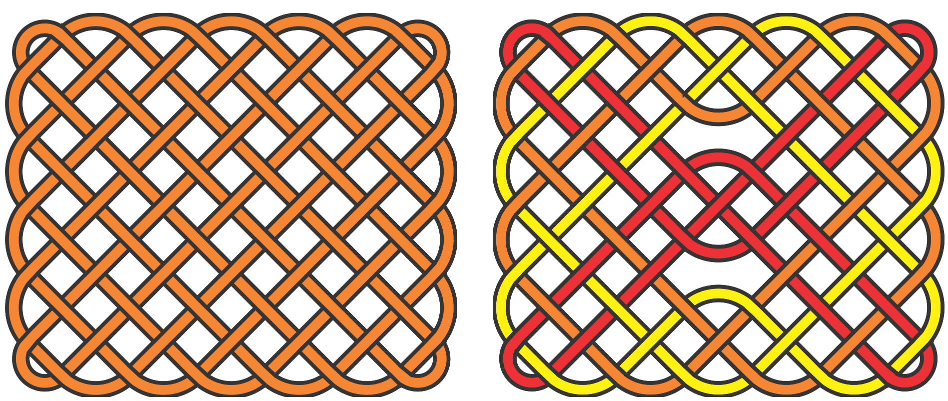

Figure 2: *Example of interlacing knot designs/diagrams - coloured*

[2] Examples of this can be found in Celtic knot designs and as far back as Coptic Egypt [5].

A similar phenomenon occurs when our group designs and plans paper-strip weavings. Our research on paper-strip weaving was originally inspired by plaited reed mats produced in Southeast Asia ([10, 11]). Basic monochrome mats are made of long, narrow strips of locally available reed or leaf, whose simple 1/1 interlacement is oriented obliquely (i.e. diagonally, or at an angle) to the exterior edges of the mat, although the local interlacement of the elements (to each other) is generally perpendicular. More elaborate plaited mats can take advantage of two kinds of decorative effects: one based on patterns in contrasting colours; the other based on patterns of in-woven (not cut) holes in the surface creating an overall lacy effect, sometimes known as "open work" [8] or "open-cut work" [10].

A guiding principle of our research has been to respect and highlight the continuity of each woven element in completing its closed circuit. This is mainly due to the fact that mat designs generally do not have any place where an element terminates as this would make the mat vulnerable to unravelling. In order to determine whether and how various original forms with in-woven holes could be woven with these closed circuits, numerous artefacts were created and diagrams were prepared. In the diagrams, each element, and thus each circuit, was identified using a distinct colour. This permitted us to readily see and trace the path of each circuit and to spot the effects of changes in the forms, the placement of holes, the types of intersections, on these paths and therefore on the colour order of the elements. For example, in the case of the in-woven *figure-8* (Figure 3, left, below), labelling colours was done in sequence from the starting point (left), using six labels. Then, after the junction[3], the additional three elements were given three more colour-labels. In the overlapped *figure-8* (Figure 3, right, where the junction consists of two superimposed, distinct plaits), six colours are sufficient to label all the elements (see [7] for a more in-depth discussion).

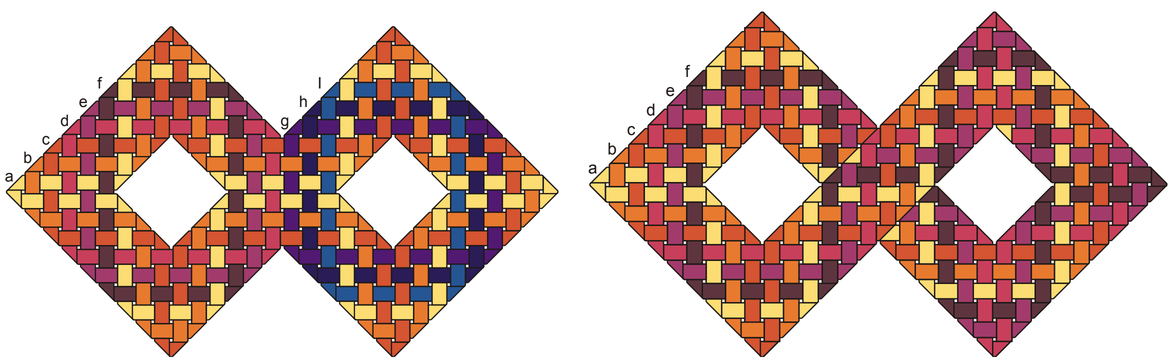

Figure 3: *Examples of paper-strip weaving diagrams using colours to signify individual paths*

This specific colouring highlights the individual elements, making it easier for the observer to take in the paths travelled by each of them. This, in turn, allows us to investigate the mathematical properties and relationships of the paths.

Some of the mathematically pertinent features of specific configurations that we have investigated using colour in this manner are:

- How many distinct circuits compose a particular woven configuration? (on the left, there are nine, on the right, six)
- Are some of the circuits global or only local to a particular region? (on the left, circuits *a*, *b* and *c* are global, the rest are local to one or the other half; on the right all six circuits are global)
- Is the order or sequence of the elements disrupted by features of the configuration? (element *b* is "between" elements *a* and *c* throughout in both artefacts; this is not always the case)
- Are there isometries between elements, that is, are there paths that are equivalent under a symmetry or transformation (on the left, elements *d* and *g* are equivalent under 180° rotation, as

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[3] We use the term junction to signify the place where multiple woven arms of the *figure-8* meet or cross [7].

are *e* and *h*, and *f* and *i*, respectively; on the right, *a* and *f* are equivalent, as are pairs *b* and *e*, and *c* and *d* respectively)?

The last feature can be highlighted using a slightly different approach than that in Figure 3, by colouring all the isometric elements (all the elements of an equivalence class) using the same colour. This further emphasises the overall symmetry of an artefact, as shown for the configuration of Figure 3, left, in Figure 4, below.

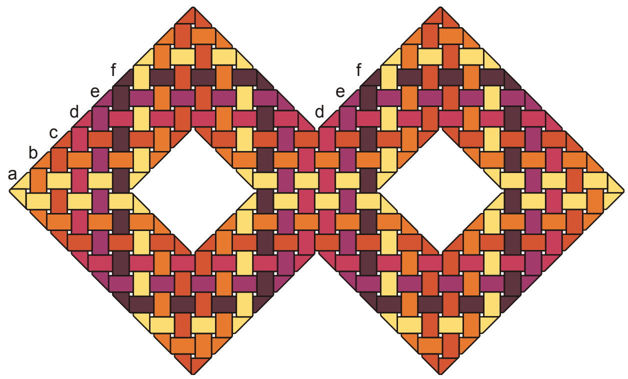

Figure 4: *Example of a paper-strip weaving diagram, using six colours to signify six isometric equivalence classes*

Figures 3 (left) and 4 emphasise different mathematical features of the same in-woven *figure-8* configuration. This selection of emphasis is an essential attribute of the meta-aesthetic of the use of colour in diagrams that we discuss here.

Working diagrams, that is, diagrams that are made as part of a design or planning process are often more rudimentary in nature than the diagrams of figures 3 and 4, which might more readily be termed *renderings*. In our work, diagrams consist of a demarcated region on a square grid (graph paper), which is then coloured using markers (if possible erasable, as errors are often made). Figure 5, below, shows such a diagram for the configuration and colouring of Figure 4.

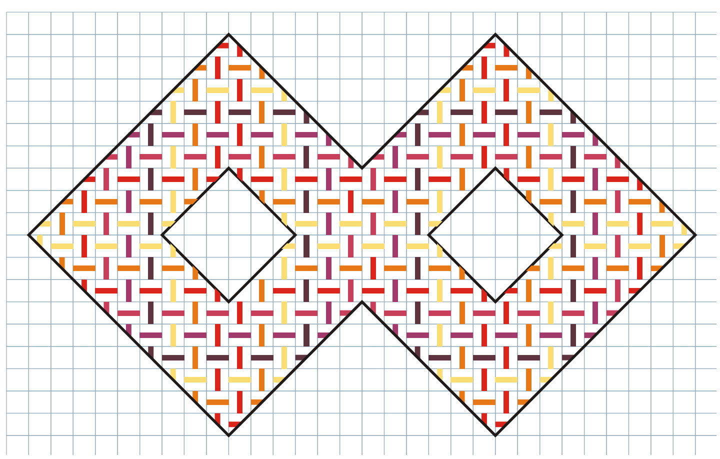

Figure 5: *Example of a rudimentary paper-strip weaving diagram, using six colours*

Working diagrams often incorporate a shorthand since they are used more as a “crib sheet” for making, than as a formal communication device, and because the user/maker is expected to “read into” the diagram, adding information that has previously been internalised, using the sensitivities we described earlier. This is a matter of degree, and sometimes a slight adjustment can make a diagram much easier to read. Figure 6, below, shows an example: using a double, instead of a single line does just enough to evoke more clearly the edges of the elements, thereby suggesting the structure more effectively.

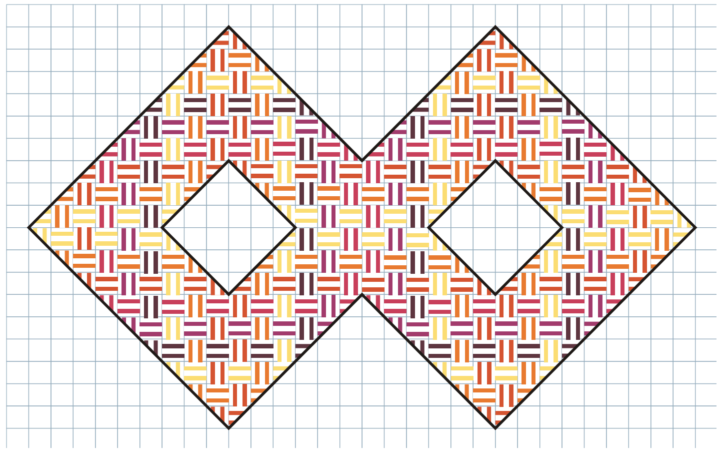

Figure 6: *Example of a clearer paper-strip weaving diagram, using six colours*

Finally, diagrams can also be used as small multiples [12] to highlight individual circuits, as in Figure 7, below, which shows a separate drawing for each circuit.

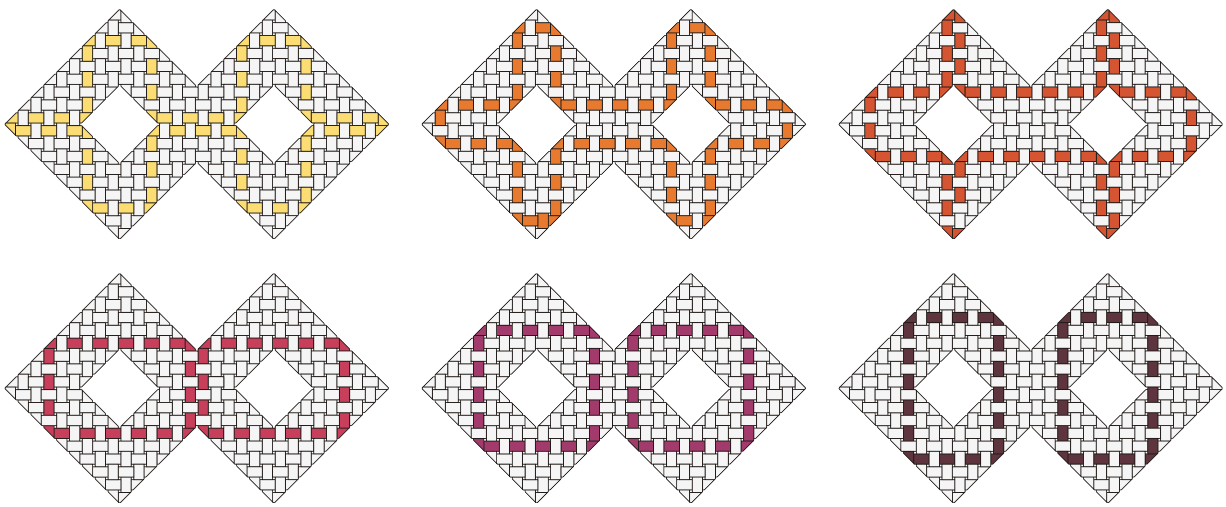

Figure 7: *Colouring individual circuits of the in-woven figure-8*

Workshop

The purpose of the workshop is to develop the sensitivity of participants to the importance and usefulness of colour by colouring interlacing knot-style diagrams, through tracing of individual circuits on a given diagram. This activity will provide opportunities to visualise, follow, and represent the way in which an element disappears and reappears as it weaves in and out of the configuration. This experience of colouring, using a variety of emphases, highlights the meta-aesthetic and mathematical similarities of structure between interlacing knots and our weaving diagrams. It also highlights mathematical concepts such as symmetry, adjacency and sequencing, which are visual design concepts as well. Drawing on participants' own perspectives, the workshop will include discussion of these ideas.

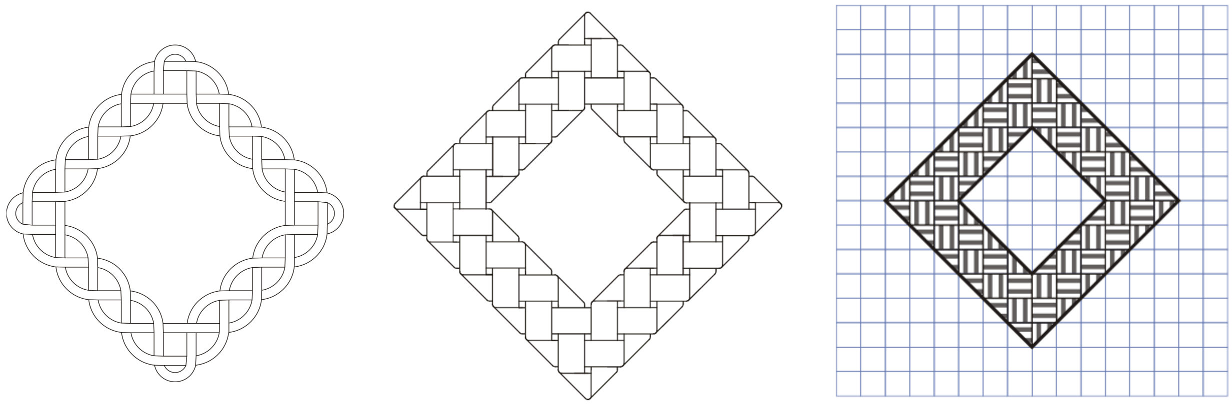

Figure 8: *Uncoloured diagrams for the 3-element, square ring*

Figure 8 exemplifies the uncoloured diagrams that will be used in this workshop. Because the interlacing knot (left) is the interpretation of the plaiting that is easiest to follow, it is a good entry point to the art of meta-aesthetic colouring. This is followed by the more stylised rendering in the centre, then the more rudimentary graph-paper diagram on the right. In the case of the 3-element, square ring, the colouring that shows individual circuits needs three colours, but the one showing isometric groups needs only two since two circuits are equivalent under (several) transformations.

Conclusion

The understanding of mathematical structures that is derived from a practice of colouring diagrams can help when evaluating possible configurations and preparing to make concrete models. More importantly, it can serve as a thinking tool to *discover* and *uncover* mathematical structure. This is salient to the identification of various kinds of relationships, which is key to mathematical understanding. Thus, diagramming is a means of developing mathematical, as well as aesthetic, sensitivities [9]. In particular, the meta-aesthetics of colour use enhances these mathematical sensitivities.

Georg Cantor said it best: *In re mathematica ars propendi quaestionem pluris facienda est quam solvendi*.[4] For us, this is true also in art, but more importantly, we see diagramming as an inherent component of this questioning. The workshop is meant as a vehicle for experiencing and discussing whether learning to make effective diagrams *is* doing math. Reflecting on diagramming may also lead to mathematical and aesthetic understandings.

Participants are provided with multiple entry points for discussion. We invite the development of a deeper understanding of the role of diagramming: as a mode of thought; as a vehicle for planning making; as a tool of enquiry and dialogue; and as a way to trace thought processes.

[4] In mathematics the art of proposing a question must be held of higher value than solving it [13].

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